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RANKS AND DEFINABILITY IN SUPERSTABLE THEORIES

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ABSTRACT
We study the notion of definable type, and use it to define the *product* of types and the *heir* of a type. Then, in the case of stable and superstable theories, we make a general study of the notion of rank. Finally, we use these techniques to give a new proof of a theorem of Lachlan on the number of isomorphism types of countable models of a superstable theory.

0. Introduction

As is well known, the ultrafilters and the complete types of a theory are both maximal filters of a Boolean algebra. Moreover, Lindström has interpreted ultrafilters over ω as complete types over a certain structure. It is tempting to generalize some of the notions which have been introduced in the study of ultrafilters in order to extend them to the study of types. A natural notion to look at is that of product, which leads to a condition on types, which we call *well-definability*. For ultrafilters over ω this condition vanishes because all of them verify it. This is where stable theories come in: every complete type over a model of a stable theory is well-definable as follows from results of Shelah.

If p is a complete definable type over a model $\mathcal{M}$, then, for any $\mathcal{A}$ which is an extension of $\mathcal{M}$, p has a priviliged extension over $\mathcal{A}$. It will be called the *heir* of p on $\mathcal{A}$. Some applications of this notion have already appeared in [9]. Yet another remarkable fact makes stable theories the natural framework for our topic: the product of complete types of stable theories commutes: this has numerous consequences. Among them a useful "reciprocity principle" (Theorem 4,7). It also leads to a general study of the notion of rank, and to the definition of a new rank U (Section 5), which in some sense is univeral and has very strong properties. As an application we give a new proof of the theorem of Lachlan [8]

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on the number of isomorphism type of countable models of a superstable theory (this theorem generalizes the well-known theorem of Baldwin and Lachlan [2]).

After this paper had been written in an initial French version the author became acquainted with the work of Shelah on *forking* which will appear in his book on stability [17]. Although this notion will not be used in this paper, let us make some remark on the relationship between our methods and those of Shelah: in Section 4, we prove that provided an ordinal-valued rank satisfies some very natural axioms, and that the theory is superstable, the relation "to have the same rank" for type $p \in S_n(\mathscr{A})$ and $p' \in S_n(\mathscr{B})$, where $\mathscr{A} \subseteq \mathscr{B}$ and $p \subseteq p'$, does not depend on the particular rank used, and we give (Proposition 4, 13) an equivalent condition which does not involve any notion of rank. If the theory is stable but not necessarily superstable, in Shelah's terminology, our condition is precisely equivalent to: p' does not fork over $\mathscr{A}$.

Shelah has proved independently most of the results for forking and stable theories which correspond to various theorems of Section 4; in particular, Theorem 4, 4 (and hence Theorem 3, 4); 4, 7; 4, 15; and Corollary 4, 16. Theorem 4, 12 can also be deduced from his results. He further proved that if $p' \in S_n(\mathscr{B})$ and $p = p' \upharpoonright \mathscr{A}$ then p' does not fork over $\mathscr{A}$ if and only if $R(p) = R(p')$, whenever R is either the Morley rank or one of the many ranks he considers. Finally our proof of Theorem 4, 12 has been simplified by the referee.

On the other hand, the notion of forking allows us to strengthen and to generalize various results of ours; statements and proofs will appear in a forthcoming paper.

1. Notations

The language L will be fixed, as well as a complete theory T in L; we shall assume that T admits elimination of quantifiers (that we may always do so, without loss of generality, is proved in [12]), and that T has no finite models. We shall sometimes also suppose that L does not contain any constant or function symbols (this will not restrict the generality of our results). L_n will denote the set of formulas whose free variable is among $v_0, v_1, \cdots, v_{n-1}$.

If A is a set, we shall denote by $\bar{A}$ the set of finite sequences from A. If $\bar{a} \in \bar{A}$ and $\bar{a} = (a_0, a_1, \cdots, a_n)$, then $|\bar{a}| = \{a_0, a_i, \cdots, a_n\}$ and $B \cup \bar{a} = B \cup |\bar{a}|$. The symbol $\bar{v}_n$ will denote the sequence $(v_0, v_1, \cdots, v_{n-1})$; $L(A)$ is the language obtained from L by adding a constant for every $a \in A$, which we shall also denote a. If $\varphi \in L$ and $\bar{a} \in \bar{A}$, $\bar{a} = (a_0, a_1, \cdots, a_n)$ then $\varphi(\bar{a})$ is the formula of $L(A)$ obtained from φ by substituting, for every i, $0 \leqq i \leqq n$, a_i for v_i.

Let $K(T)$ be the category whose objects are substructures of models of T and whose maps are monomorphisms between such objects (see [14] p. 165). The letters $\mathcal{A}, \mathcal{A}', \mathcal{B}, \cdots$ will denote objects of $K(T)$, and $\mathcal{M}, \mathcal{M}', \cdots$, will always denote models of $T; A, A', B, \cdots$ $M, M', \cdots$ are the universes of $\mathcal{A}, \mathcal{A}', \mathcal{B}, \cdots \mathcal{M}, \mathcal{M}', \cdots$. Since we have elimination of quantifiers the set

$$\{\varphi(\bar{a});\ \varphi \in L,\ \bar{a} \in \bar{A},\ \mathcal{M} \models \varphi(\bar{a})\}$$

does not depend on $\mathcal{M}$, provided that $\mathcal{A} \subseteq \mathcal{M}$, and will be denoted by $T(\mathcal{A})$.

Let $\{x_0, x_1, \cdots, x_n, \cdots\}$ be a set of new individual constants which we distinguish from the individual variables, and set $\bar{x}_n = (x_0, x_1, \cdots, x_{n-1})$. An n-type over $\mathcal{A}$ is a set of formulas of $L(A \cup \bar{x}_n)$ consistent with $T(\mathcal{A})$. A complete n-type over $\mathcal{A}$ is a complete theory in $L(A \cup \bar{x}_n)$ which extends $T(\mathcal{A})$; $S_n(\mathcal{A})$ is the set of all complete types over $\mathcal{A}$; and $S_n(\emptyset) = S_n(T)$ whenever $\emptyset \in K(T)$.

The set $S_n(\mathcal{A})$ will be made into a topological space in the usual way (see [11]); if f is monomorphism from $\mathcal{A}$ into $\mathcal{B}$, we shall denote by $\hat{f}^n$ the corresponding continuous map from $S_n(\mathcal{B})$ onto $S_n(\mathcal{A})$. More precisely, if $p \in S_n(\mathcal{B})$,

$$\hat{f}^n(p) = \{\varphi(a_0, a_1, \cdots, a_{m-1});\ \varphi \in L(\bar{x}_n),\ (a_0, a_1, \cdots, a_{m-1}) \in \bar{A},$$
$$\varphi(f(a_0),\ f(a_1), \cdots, f(a_{m-1})) \in p\}.$$

When $\mathcal{A} \subseteq \mathcal{B}$, $e_{\mathcal{A},\mathcal{B}}$ is the canonical injection from $\mathcal{A}$ into $\mathcal{B}$; we set $i^n_{\mathcal{B},\mathcal{A}} = \hat{e}^n_{\mathcal{A},\mathcal{B}}$.

If $\mathcal{A} \subseteq \mathcal{B}$ and $\bar{b} \in \bar{M}$, the type realized by $\bar{b}$ over $\mathcal{A}$ in $\mathcal{M}$, denoted by $t_{\mathcal{M}}(\bar{b}, \mathcal{A})$, is defined by

$$t_{\mathcal{M}}(\bar{b}, \mathcal{A}) = \{\varphi(\bar{x}_n);\ \varphi \in L_n(A) \text{ and } \mathcal{M} \models \varphi(\bar{b})\}.$$

It is clear that, if $\mathcal{M} \subseteq \mathcal{M}'$, $t_{\mathcal{M}}(\bar{b}, \mathcal{A}) = t_{\mathcal{M}}(\bar{b}, \mathcal{A})$, so that, context permitting, we shall write $t(\bar{b}, \mathcal{A})$ instead of $t_{\mathcal{M}}(\bar{b}, \mathcal{A})$.

2. Definable types

Definition 1. We say that d is an *n-preschema* on $\mathcal{A}$ if d is a map from $L(\bar{x}_n)$ into $L(A)$ such that for all $k \in \omega$, if $\varphi \in L_k(\bar{x}_n)$, then $d(\varphi) \in L_k(A)$.

Let $\mathcal{B} \supseteq \mathcal{A}$, and consider the following set:

$$d(\mathcal{B}) = \{\varphi(\bar{b});\ \varphi \in L_k(x_n),\ k \in \omega,\ \bar{b} \in B^k, \text{ and } d(\varphi) \in T(\mathcal{B})\}.$$

If this set is an n-type over $\mathcal{B}$, we shall say that $d(\mathcal{B})$ is *defined* by d over $\mathcal{B}$.

Definition 2. Let $p \in S_n(\mathcal{B})$, and $\mathcal{A} \subseteq \mathcal{B}$. We say that p is *definable* on $\mathcal{A}$ if and only if there is an n-preschema on $\mathcal{A}$ which defines p over $\mathcal{B}$; p is definable if it is definable on $\mathcal{B}$.

PROPOSITION 3. *Let $\mathcal{A} \subseteq \mathcal{M}$ and d be an n-preschema on $\mathcal{A}$. If $d(\mathcal{M})$ is an n-type over $\mathcal{M}$ which includes $T(\mathcal{M})$ and is deductively closed, then the three following conditions hold:*

1) *If $\varphi \in L$, then $T(\mathcal{A}) \vdash \varphi \leftrightarrow d(\varphi)$.*
2) *If $\varphi, \varphi' \in L(\bar{x}_n)$ and $T \vdash \varphi \to \varphi'$, then $T(\mathcal{A}) \vdash d(\varphi) \to d(\varphi')$.*
3) *If φ, $\varphi' \in L(\bar{x}_n)$ then $T(\mathcal{A}) \vdash d(\varphi \to \varphi') \to (d(\varphi) \to d(\varphi'))$.*

If, moreover, $d(\mathcal{M})$ is a complete n-type, then conditions 4) *and* 5) *are also satisfied:*

4) *If $\varphi \in L(x_n)$ then $T(\mathcal{A}) \vdash \neg d(\varphi) \leftrightarrow d(\neg \varphi)$.*
5) *If $\varphi, \varphi' \in L(\bar{x}_n)$ then $T(\mathcal{A}) \vdash d(\varphi \wedge \varphi') \leftrightarrow (d(\varphi) \wedge d(\varphi'))$ and $T(\mathcal{A}) \vdash d(\varphi \vee \varphi') \leftrightarrow (d(\varphi) \vee d(\varphi'))$.*

PROOF. 1) Suppose that $\varphi \in L_k$ and let $\bar{b} \in M^k$; then $\varphi(\bar{b}) \in T(\mathcal{M})$ if and only if $\varphi(\bar{b}) \in d(\mathcal{M})$, if and only if $d(\varphi)(\bar{b}) \in T(\mathcal{M})$. So

$$\mathcal{M} \models \forall \bar{v}_k (\varphi \leftrightarrow d(\varphi)).$$

But this formula belongs to $L(A)$, and that implies:

$$T(\mathcal{A}) \vdash \varphi \leftrightarrow d(\varphi).$$

2) Suppose now that $\varphi, \varphi' \in L_k(\bar{x}_n)$ and

$$T \vdash \varphi \to \varphi'$$

and let $\bar{b} \in M^k$. If

$$\mathcal{M} \models d(\varphi)(\bar{b})$$

then $\varphi(\bar{b}) \in d(\mathcal{M})$, and since $d(\mathcal{M})$ includes T and is deductively closed $\varphi'(\bar{b}) \in d(\mathcal{M})$, and

$$\mathcal{M} \models d(\varphi')(\bar{b}).$$

So

$$\mathcal{M} \models \forall \bar{v}_k (d(\varphi) \to d(\varphi'))$$

and

$$T(\mathcal{A}) \vdash d(\varphi) \to d(\varphi').$$

Conditions 3), 4), and 5) are proved in the same way.

DEFINITION 4. An *n-schema* on $\mathcal{A}$ is an n-preschema on $\mathcal{A}$ verifying conditions 1), 2), 3), 4).

If $p \in S_n(\mathcal{B})$, $\mathcal{B} \supseteq \mathcal{A}$ is defined by an n-schema on $\mathcal{A}$, then p is *well-definable over* $\mathcal{A}$; p is *well definable* if it is well definable over $\mathcal{B}$.

PROPOSITION 5. *Let d be an n-preschema on $\mathcal{A}$ satisfying conditions* 1), 2) *and* 3), *and $\mathcal{B} \supseteq \mathcal{A}$. Then $d(\mathcal{B})$ is an n-type over $\mathcal{B}$. If d is an n-schema, then $d(\mathcal{B}) \in S_n(\mathcal{B})$.*

PROOF. Suppose d satisfies conditions 1), 2) and 3). First we prove that $d(\mathcal{B})$ is closed under deduction: suppose that $\varphi, \varphi' \in L_k(\bar{x}_n), \bar{b} \in B^k$ and $\varphi(\bar{b}) \to \varphi'(\bar{b}) \in d(\mathcal{B})$ and $\varphi(\bar{b}) \in d(\mathcal{B})$; we know that

$$d(\varphi \to \varphi')(\bar{b}), d(\varphi)(\bar{b}) \in T(\mathcal{B})$$

and from condition 3)

$$\forall \bar{v}_k (d(\varphi \to \varphi') \to (d(\varphi) \to d(\varphi')) \in T(\mathcal{B}).$$

Since $T(\mathcal{B})$ itself is closed under deduction:

$$d(\varphi')(\bar{b}) \in T(\mathcal{B})$$

and

$$\varphi'(\bar{b}) \in d(\mathcal{B}).$$

Suppose now that $\varphi \in L_{k+1}(\bar{x}_n)$, $\bar{b} \in B^k$, and

$$\forall v_0 \varphi(v_0, \bar{b}) \in d(\mathcal{B}).$$

Let $b' \in \mathcal{B}$; clearly

$$\vdash \forall v_0 \varphi \to \varphi$$

and by condition 2)

$$\forall \bar{v}_{k+1}(d(\forall v_0 \varphi) \to d(\varphi)) \in T(\mathcal{B}).$$

Since $d(\forall v_0 \varphi(v_0, \bar{b})) \in T(\mathcal{B})$, $d(\varphi)(b', \bar{b}) \in T(\mathcal{B})$ and $\varphi(b', \bar{b}) \in d(\mathcal{B})$.

To see why $d(\mathcal{B})$ is consistent, it is sufficient to notice that $\exists v_0(v_0 \neq v_0) \notin d(\mathcal{B})$, since from 1) $d(\exists v_0(v_0 \neq v_0) \leftrightarrow \exists v_0(v_0 \neq v_0) \in T(\mathcal{B})$ and $\exists v_0(v_0 \neq v_0) \notin T(\mathcal{B})$.

Now it is clear from condition 1) that $T(\mathcal{B}) \subseteq d(\mathcal{B})$, so we have proved that $d(\mathcal{B})$ is an n-type over $\mathcal{B}$.

Suppose now that condition 4) is satisfied, $\varphi \in L_k(\bar{x}_n), \bar{b} \in B_k$ and $\varphi(\bar{b}) \notin d(\mathcal{B})$. Then $d(\varphi)(\bar{b}) \notin T(\mathcal{B})$, and $\neg\, d(\varphi)(\bar{b}) \in T(\mathcal{B})$. So $d(\neg \varphi)(\bar{b}) \in T(\mathcal{B})$, and $\neg \varphi(\bar{b}) \in d(\mathcal{B})$.

COROLLARY 6. *If d is an n-schema on $\mathcal{A}$, then d satisfies condition 5).*

PROOF. Indeed, let $\mathcal{M} \supseteq \mathcal{A}$; then $d(\mathcal{M}) \in S_n(\mathcal{A})$, and from Proposition 3 d satisfies Condition 5).

PROPOSITION 7. *Let d and d' be two schemata on $\mathcal{M}$. Then the three following conditions are equivalent:*

1) $d(\mathcal{M}) = d'(\mathcal{M})$.
2) *For all* $\varphi \in L(\bar{x}_n)$, $\mathcal{M} \models d(\varphi) \leftrightarrow d'(\varphi)$.
3) *For all* $\mathcal{A} \supseteq \mathcal{M}$, $d(\mathcal{A}) = d'(\mathcal{A})$.

PROOF.
1)→2): If $\varphi \in L_k(\bar{x}_n)$ and $\bar{b} \in M^k$, then

$$\mathcal{M} \models d(\varphi)(\bar{b}) \text{ if and only if } \mathcal{M} \models d'(\varphi)(\bar{b}).$$

So $\mathcal{M} \models \forall \bar{v}_k (d(\varphi) \leftrightarrow d'(\varphi))$ and

$$\mathcal{M} \models d(\varphi) \leftrightarrow d'(\varphi).$$

2)→3): If $\varphi \in L_k(\bar{x}_n)$, and $\bar{a} \in A^k$, it is clear that $d(\varphi)(\bar{a}) \in T(\mathcal{A})$ if and only if $d'(\varphi)(\bar{a}) \in T(\mathcal{A})$. So $d(\mathcal{A}) = d'(\mathcal{A})$.

3)→1) is obvious.

DEFINITION 8. Let p be a definable complete n-type over $\mathcal{M}$, and $\mathcal{M} \subseteq \mathcal{A}$. The *heir* of p on $\mathcal{A}$ is the type $d(\mathcal{A})$, where d is any schema on $\mathcal{M}$ defining p.

From Proposition 3, we see that in fact p is well-definable, and clearly $d(\mathcal{A})$ does not depend on d, provided that $d(\mathcal{M}) = p$. Of course $d(\mathcal{A})$ is an extension of p.

The proof of the following proposition is an easy exercise:

PROPOSITION 9.

1) *Let $\mathcal{M} \supseteq \mathcal{A} \supseteq \mathcal{B}$, and p be a definable complete type over $\mathcal{M}$. Then the heir of p on $\mathcal{B}$ is an extension of the heir of p on $\mathcal{A}$.*

2) *If $\mathcal{M} \subseteq \mathcal{A}$, $p \in S_n(\mathcal{A})$, and p is definable on $\mathcal{M}$, then p is the heir of $p \restriction \mathcal{M}$.*

3) *If $\mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{A}$, and p is a definable complete type over $\mathcal{M}$, then the heir of p on $\mathcal{A}$ is the heir on $\mathcal{A}$ of the heir of p on $\mathcal{M}_1$.*

4) *Let f be an isomorphism from $\mathcal{A}$ onto $\mathcal{A}'$, $\mathcal{M} \subseteq \mathcal{A}$, $\mathcal{M}' \subseteq \mathcal{A}'$, such that $M = f(M')$, and $p \in S_n(\mathcal{A})$ such that p is the heir of $p \restriction \mathcal{M}$. Then $\hat{f}(p)$ is the heir of $\hat{f}(p) \restriction \mathcal{M}$.*

5) *Let $\mathcal{M} \subseteq \mathcal{A} \subseteq \mathcal{B}$, $\bar{b} = (b_0, b_1, \cdots, b_{n-1}) \in B^n$, τ be a map of $l \in \omega$ into n,*

$\bar{b}' = (b_{\tau(0)}, b_{\tau(1)}, \cdots, b_{\tau(l-1)})$. *If* $t(\bar{b}, \mathscr{A})$ *is the heir of* $t(\bar{b}, \mathscr{M})$, *then* $t(\bar{b}', \mathscr{A})$ *is the heir of* $t(\bar{b}', \mathscr{M})$.

We shall now illustrate these notions by examples.

EXAMPLE 1 (ultrafilters). As far as the author knows, the connection between ultrafilters and complete types first appears in [10]. We shall briefly recall the essential facts.

Let L be the language containing, for all $n \in \omega$ an individual constant symbol $\underset{\sim}{n}$, for all $n \in \omega$ and $A \subseteq \omega^n$ an n-placed predicate symbol $\underset{\sim}{A}$, and for all $n \in \omega$, and any map f from ω^n into ω an n-placed function symbol $\underset{\sim}{f}$; $\mathscr{N}$ will be the L-structure whose universe is ω and where $\underset{\sim}{n}$ is interpreted by n, $\underset{\sim}{A}$ by A, and $\underset{\sim}{f}$ by f; let T_0 be the theory of $\mathscr{N}$. Then T_0 admits elimination of quantifiers and is universal.

Let $p \in S_n(\mathscr{N})$, and consider

$$\alpha_n(p) = \{A\,; A \subseteq \omega^n \text{ and } \underset{\sim}{A}(\bar{x}_n) \in p\}.$$

It is not difficult to see that $\alpha_n(p)$ is an ultrafilter over ω^n. Let $\beta(\omega^n)$ be the topological space whose universe is the set of all ultrafilters over ω^n, with $C = \{\{p\,; A \in p\}; A \subseteq \omega^n\}$ a basis for closed sets. We see that $\beta(\omega^n)$ is the Stone-Cech compactification of ω^n, if ω^n has the discrete topology. It is a compact Hausdorff space, and C is also a basis for the open sets.

It is easily checked that α_n is a one-one continuous map from $S_n(\mathscr{N})$ onto $\beta(\omega^n)$.

Let $\mathscr{M} \subseteq \mathscr{M}'$ be models of T_0 and $\bar{a} \in \bar{M}'$. We denote by $\mathscr{M}(\bar{a})$ the structure generated by $\mathscr{M}$ and $\bar{a}$ in $\mathscr{M}'$. The following result appears in [9].

PROPOSITION 10. *Let* $\mathscr{N} < \mathscr{M}$, *and* $\bar{a} \in M^n$. *Then* $\mathscr{N}(\bar{a})$ *is isomorphic to* $\mathscr{N}^{\omega^n}/p$, *if* $p = \alpha_n(t(\bar{a}, \mathscr{N}))$.

We now exhibit a link between ultrafilters and definability.

PROPOSITION 11. *Let* $p \in S_n(\mathscr{N})$; *then* p *is definable.*

PROOF. For every $k \in \omega$ and $\varphi \in L_k(\bar{x}_n)$, set

$$A(\varphi) = \{\bar{a}\,; \bar{a} \in \omega^k \quad \text{and} \quad \varphi(\bar{a}) \in p\},$$

and $d(\varphi) = \underset{\sim}{A}(\varphi)(\bar{v}_k)$. It is clear that d defines p over $\mathscr{N}$.

It will be obvious from Proposition 22 below that it is impossible that for every $\mathscr{M}$, model of T, and $p \in S_1(\mathscr{M})$, p be definable. But we can use the following result, due to Ressayre, to prove something stronger:

PROPOSITION 12. *Let $\mathcal{M} \subseteq \mathcal{M}'$ be a model of T_0, $a \in M' - M$, and suppose that $t(a, \mathcal{M})$ is definable. Then $\mathcal{M}$ (a) is an end-extension of $\mathcal{M}$ (that is, for all $b \in M(a) - M$ and $c \in M$, $\mathcal{M}(a) \models c \leqq b$; the interpretation of $\leqq$ in $\mathcal{N}$ is just the natural ordering).*

PROOF. Suppose $b \in M(a)$, $c \in M'$, and

$$\mathcal{M}(a) \models b \leqq c.$$

There exist $n \in \omega$, i a map from ω^n on ω, and $\bar{d} \in M^{n-1}$ such that:

$$\mathcal{M}(a) \models b = \underline{i}(\bar{d}, a).$$

Since $t(a, \mathcal{M})$ is definable, there is $\psi \in L_1(M)$ such that, for all $m \in M$

$$\mathcal{M}(a) \models \underline{i}(\bar{d}, a) \leqq m \quad \text{if and only if} \quad \mathcal{M} \models \psi(m).$$

The Peano axioms are in T_0, so there is c_0 which is the least element of $\mathcal{M}$ verifying $\psi(v_0)$. Then:

$$\mathcal{M}(a) \models b \leqq c_0 \wedge \neg(b \leqq c_0 - 1).$$

So $b = c_0$ and $b \in M$.

COROLLARY 13. *If $\mathcal{N} < \mathcal{M}$, $\mathcal{N} \neq \mathcal{M}$, then there is $p \in S_1(\mathcal{M})$ which is not definable.*

PROOF. Let $a \in M - N$; the following set of formulas:

$$\{x_0 \leqq a\} \cup \{x_0 \neq m\,;\, m \in M\} \cup T(\mathcal{M})$$

is consistent. Let $\mathcal{M}'$ be an extension of $\mathcal{M}$ containing an element b verifying all these formulas. Then $\mathcal{M}(b)$ is not an end extension of $\mathcal{M}$ and $t(b, \mathcal{M})$ is not definable.

The literature concerning $\beta(\omega)$ is plentiful. We shall see that we can interpret some notions which have been introduced over $\beta(\omega)$ in model theory using the notion of definability.

If F and G are ultrafilters over ω, set

$$F \times G = \{A\,;\, A \subseteq \omega^2 \quad \text{and} \quad \{n\,;\{m\,;(n, m) \in A\} \in F\} \in G\}.$$

It is well known (see [4], for example) that $F \times G$ is an ultrafilter over ω^2, and we have:

PROPOSITION 14. *Let p and q belong to $S_1(\mathcal{N})$, $\mathcal{N} < \mathcal{M}$, $b \in M$ realizing p and q respectively over $\mathcal{N}$. Then*

$$\alpha_2(t(a,b),\mathcal{N}) = \alpha_1(t(a,\mathcal{N})) \times \alpha_1(t(b,\mathcal{N})) = \alpha_1(p) \times \alpha_1(q)$$

if and only if $t(a,\mathcal{N}(b))$ *is the heir of* $t(a,\mathcal{N})$ *on* $\mathcal{N}(b)$.

PROOF. First, suppose that $t(a,\mathcal{N}(b))$ is the heir of $t(a,\mathcal{N})$ on $\mathcal{N}(b)$. Let $A \subseteq \omega^2$, and set

$$B = \{n\,; A(x_0, n) \in p\}.$$

If d is a schema on $\mathcal{N}$ defining p, then

$$\mathcal{N} \models d(\underset{\sim}{A}(x_0, v_0)) \leftrightarrow \underset{\sim}{B}(v_0)$$

and the following are equivalent

$$\underset{\sim}{A}(x_0, x_1) \in t((a,b),\mathcal{N})$$

$$\underset{\sim}{A}(x_0, b) \in t(a,\mathcal{N}(b))$$

$$\mathcal{M} \models \underset{\sim}{B}(b)$$

$$B \in \alpha_1(q).$$

So $A \in \alpha_2(t(a,b),\mathcal{N})$ if and only if $\{n\,; \underset{\sim}{A}(x_0,n) \in p\} \in \alpha_1(q)$. But, for every $n \in \omega$, $\underset{\sim}{A}(x_0,n) \in p$ if and only if $\{m\,;(n,m) \in A\} \in \alpha_1(p)$ and we are done.

Suppose now that $\alpha_2(t(a,b),\mathcal{N})) = \alpha_1(p) \times \alpha_2(q)$. Let $\mathcal{M}' \supseteq \mathcal{M}$, and $a' \in M'$ such that $t(a',\mathcal{N}(b))$ is the heir of p on $\mathcal{N}(b)$. Then, from the first part, $\alpha_2(t((a,b),\mathcal{N})) = \alpha_2(t((a',b),\mathcal{N}))$, and since α_2 is one-one, $t((a,b),\mathcal{N}) = t((a',b),\mathcal{N})$ and also $t(a,\mathcal{N}(b)) = t(a',\mathcal{N}(b))$.

Of particular importance are also the Rudin-Keisler and the Rudin-Frolik order.

Let $F \in \beta(\omega)$, and i a map from ω into ω. Then the set

$$\{A\,; A \subseteq \omega \quad \text{and} \quad i^{-1}(A) \in F\}$$

is an ultrafilter over ω, which we shall denote by $\hat{\imath}(F)$. For $F, G \in \beta(\omega)$ we say $F \underset{RK}{\geqq} G$ if and only if there is a map i from ω on ω, such that $G = \hat{\imath}(F)$. The following is easily proved:

PROPOSITION 15. *Let p and q belong to $S_1(\mathcal{N})$. Then $\alpha_1(p) \underset{RK}{\geqq} \alpha_2(q)$ if and only if every extension of $\mathcal{N}$ realizing p realizes q.*

DEFINITION 16. Let $(F_i\,; i \in \omega)$ be a sequence of ultrafilters over ω; we say that this sequence is *discrete* if there is a sequence $(A_i; i \in \omega)$ of pairwise disjoint subsets of ω such that, for all $i \in \omega$, $A_i \in F_i$.

If $(F_i; i \in \omega)$ is a sequence of ultrafilters over ω, and $G \in \beta(\omega)$, then

$$\{A; A \subseteq \omega \quad \text{and} \quad \{i; A \in F_i\} \in G\}$$

is an ultrafilter over ω. It is the limit of $(F_i; i \in \omega)$ along G. If $(F_i; i \in \omega)$ is discrete we shall denote that ultrafilter by $\Sigma[(F_i; i \in \omega), G]$.

Definition 17. We say that $F \underset{RF}{\geqq} G$ if there is a discrete sequence $(F_i; i \in \omega)$ such that $F = \Sigma[(F_i; i \in \omega), G]$.

It is proved in [4], that the relation $\underset{RF}{\geqq}$ is a preordering and that $F \underset{RF}{\geqq} G$ implies $F \underset{RK}{\geqq} G$. Now we have:

Proposition 18. *Let $p, q \in S_1(\mathcal{N})$, and suppose that $\alpha_1(p) \underset{RF}{\geqq} \alpha_1(q)$; then there exist $\mathcal{M} \supseteq \mathcal{N}$, $a \in M$ realizing p over $\mathcal{N}$, $b \in N(a)$ realizing q over $\mathcal{N}$, such that $t(a, \mathcal{N}(b))$ is definable.*[†]

Proof. Let $(F_i; i \in \omega)$ be a discrete sequence of ultrafilters over ω such that $\alpha_1(p) = \Sigma[(F_i; i \in \omega), \alpha_1(q)]$, and let $(A_i; i \in \omega)$ be a sequence of pairwise disjoint subsets of ω, such that for all $i \in \omega$, $A_i \in F_i$. We may suppose that $\{A_i; i \in \omega\}$ is a partition of ω. Let h be the map from ω into ω such that, for $i \in \omega$, $i \in A_{h(i)}$, and let $\mathcal{M}$ be an extension of $\mathcal{N}$ and $a \in M$ realize p over $\mathcal{N}$.

Claim 1: If b is the element of $N(a)$ such that

$$\mathcal{M} \models b = \underset{\sim}{h}(a),$$

then $t(b, \mathcal{N}) = q$. Indeed, $\underset{\sim}{A}(x_0) \in q$ if and only if $A \in \alpha_1(q)$, if and only if $\bigcup_{i \in A} A_i \in \alpha_1(p)$; but $\bigcup_{i \in A} A_i = h^{-1}(A)$. So $\underset{\sim}{A}(x_0) \in q$ if and only if

$$\mathcal{M} \models \underline{h^{-1}(A)}(a)$$

and it is clear that this is equivalent to

$$\mathcal{M} \models \underset{\sim}{A}(\underset{\sim}{h}(a))$$

and to

$$\mathcal{M} \models \underset{\sim}{A}(b).$$

[†] A. Blass has proved that the converse is not true.

Define now d, a preschema on $\mathcal{N}(b)$: for all $n \in \omega$, and $\varphi \in L_{n+1}$

$$d(\varphi(x_0, \bar{v}_n)) = \underset{\sim}{A}(\varphi)(b, \bar{v}_n)$$

where $A(\varphi) = \{(i, \bar{y}); i \in \omega, \bar{y} \in \omega^n \text{ and } \{n; \mathcal{N} \models \varphi(n, \bar{y})\} \in F_i\}$.

One can check that d is a 1-schema on $\mathcal{N}(b)$.

Claim 2: $d(\mathcal{N}(b))$ is an extension of p; for all $B \subseteq \omega$, $\underset{\sim}{B}(x_0) \in p$ if and only if

$$A = \{i; i \in \omega \quad \text{and} \quad \{n; n \in \omega \quad \text{and} \quad \mathcal{N} \models \underset{\sim}{B}(n)\} \in F_i\} \in \alpha_1(q).$$

But $\underset{\sim}{A}(b) = d(\underset{\sim}{B}(x_0))$, and so $\underset{\sim}{B}(x_0) \in p$ if and only if $\underset{\sim}{B}(x_0) \in d(\mathcal{N}(b))$.

Claim 3: $\underset{\sim}{h}(x_0) = b \in d(\mathcal{N}(b))$. We have

$$d(\underset{\sim}{h}(x_0) = v_0) = \underset{\sim}{A}(b, v_0), \quad \text{where}$$

$$A = \{(i, y); (i, y) \in \omega^2 \quad \text{and} \quad \{n; n \in \omega \quad \text{and} \quad \mathcal{N} \models \underset{\sim}{h}(n) = y\} \in F_i\}.$$

But $\{n; n \in \omega \text{ and } \mathcal{N} \models \underset{\sim}{h}(n) = y\} = A_y$, and $A_y \in F_i$ if and only if $y = i$. So

$$\mathcal{M} \models d(\underset{\sim}{h}(x_0) = v_0)) \leftrightarrow b = v_0$$

and $\underset{\sim}{h}(x_0) = b \in d(\mathcal{N}(b))$.

Now let $\mathcal{M}'$ be an extension of $\mathcal{N}(b)$ generated by $\mathcal{N}(b)$ and $a' \in M'$, where $t(a', \mathcal{N}(b)) = d(\mathcal{N}(b))$; then $\mathcal{M}'$ is generated by a' alone, $\mathcal{M}' = \mathcal{N}(a')$, and $\mathcal{M}'$ is isomorphic to $\mathcal{N}(a)$.

EXAMPLE 2 (abelian groups). Let L be the language whose similarity type is $(0, +, -, =, R_n, n \geqq 1)$, where 0 is a constant symbol, $+$, $-$ are binary function symbols, and each R_n is a 1-placed predicate symbol. By a group we mean an L-structure satisfying the axioms of abelian groups and the formulas

$$\forall v_0(R_n(v_0) \leftrightarrow \exists v_1(nv_1 = v_0)) \quad \text{for} \quad n \geqq 1.$$

We shall fix an infinite group G, and let T_1 be the theory of G. It is proved in [19] that

PROPOSITION 19. *T_1 admits elimination of quantifiers.*

From this we can deduce (see [3]):

PROPOSITION 20. *For all models $\mathcal{M}$ of T_1, and $p \in S_n(\mathcal{M})$, p is definable.*

A consequence of this and of Proposition 22, below, is that T_1 is stable.

Suppose now that G is a torsion-free group; therefore every model of T is a torsion-free group. Let $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \mathcal{M}_2$ be models of T_1, and $a \in M_2$. We shall denote by $\mathcal{M}_i(a)$ $(i = 0, 1)$ the pure subgroup generated by $\mathcal{M}_i$ and a. We have:

$M_i(a) = \{m; m \in M_2$ and there exist $n \neq 0, k \in \omega, b \in M_i$ such that $nm = ka + b\}$.

We leave the following as an exercise for the reader:

Claim 1: If $a \in M_2 - M_1$, then $t(a, \mathcal{M}_1)$ is the heir of $t(a, M_0)$ if and only if $\mathcal{M}_1(a) = \mathcal{M}_0(a) + \mathcal{M}_1$.

This can also be expressed by

Claim 2: If $a \in M_2$, then $t(a, \mathcal{M}_1)$ is the heir of $t(a, \mathcal{M}_0)$ if and only if

$$\mathcal{M}_1(a)/\mathcal{M}_0 = \mathcal{M}_0(a)/\mathcal{M}_0 \oplus \mathcal{M}_1/\mathcal{M}_0.$$

EXAMPLE 3. Let p be an isolated complete n-type on $\mathcal{A}$; then p is definable.

Let $\psi(\bar{x}_n)$, where $\psi \in L_n(A)$, be such that p is the unique complete n-type on $\mathcal{A}$ containing $\psi(\bar{x}_n)$. Then, for every $k \in \omega$, $\varphi \in L_{n+k}$, and $\bar{a} \in A^k$, we have

$$\varphi(\bar{x}_n, \bar{a}) \in p \text{ if and only if } \forall \bar{v}_n(\psi(\bar{v}_n) \rightarrow \varphi(\bar{v}_n, a)) \in T(\mathcal{A}).$$

So if we set

$$d(\varphi(\bar{x}_n, \bar{v}_k)) = \forall \bar{w}_n(\psi(\bar{w}_n) \rightarrow \varphi(\bar{w}_n, \bar{v}_k))$$

d defines p on $\mathcal{A}$.

EXAMPLE 4. There are complete types which are definable but not well-definable. For example, let T be the theory of algebraically closed fields of characteristic 0, **C** be the field of complex numbers, **Q** the field of rational numbers, and consider p, the unique complete 1-type over **Q** containing $x_0^2 = 2$. This type is definable, from what we have said in Example 3.

Suppose that there is a schema on **Q**, say d, which defines p. Then one and only one of the formulas $x_0 = \sqrt{2}$ and $x_0 = -\sqrt{2}$ belongs to $d(\mathbf{C})$, so one and only one of the formulas $d(x_0 = v_0)(\sqrt{2}$ or $d(x_0 = v_0)(-\sqrt{2})$ is true in **C**, and this is impossible since $d(x_0 = v_0) \in L(\mathbf{Q})$, and $\sqrt{2}$ and $-\sqrt{2}$ realizes the same type over **Q**.

If T is $\aleph_0$-stable, it is possible to characterize the well-definable types: they are those whose degree (as defined in [12]) is 1. Shelah ([17]) has proved a more general result which characterizes these types for any stable theory.

We go back now to the general theory. Everything here rests on the following theorem, which has been proved by Shelah ([16]), and Baldwin ([1]):

THEOREM 21. *If T is stable, then every complete type on $\mathcal{A}$ is definable.*

This theorem admits a converse ([16]):

PROPOSITION 22. *If, for every $\mathcal{A} \in K(T)$ every complete type on $\mathcal{A}$ is definable, then T is stable.*

PROOF. If $\| A \| = \lambda$, there are no more than $\lambda^{\|T\|}$ maps from L into $L(A)$.

In [9] there is an application of the notion of heir to two-cardinal problems in countable stable theories.

3. Product of types

In this section, we suppose that T does not contain any function symbol.

DEFINITION 1. Let $p \in S_n(\mathcal{M})$, p definable, $q \in S_l(\mathcal{M})$. We define the product $p \times q$ as the $n + l$ complete type on $\mathcal{M}$ realized by $(\bar{a} \,^\wedge \bar{b})$, where:

- $\bar{b} \in M'^l$, $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{M}'$ is $(\| M \|)^+$-saturated and $t(\bar{b}, \mathcal{M}) = q$.
- $\bar{a} \in M'^n$, and $t(\bar{a}, \mathcal{M} \cup \bar{b})$ is the heir of p on $\mathcal{M} \cup \bar{b}$.

We should stress, of course, that the definition is coherent, i.e. provided the conditions above are satisfied, $t((\bar{a} \,^\wedge \bar{b}), \mathcal{M})$ does not depend on the particular $\bar{a}$, $\bar{b}$ and $\mathcal{M}'$ chosen. It is possible to define $p \times q$ in a purely syntactical way, but this leads to complicated notations.

Examples 1 and 2 of Section 2 illustrate the notion of products of types. In particular, the product of ultrafilters corresponds to the product of types. The two following propositions are easily proved:

PROPOSITION 2. *Let $p \in S_n(\mathcal{M})$, p definable, $\mathcal{M} \supseteq \mathcal{M}$, p' the heir of p on $\mathcal{M}'$, $q \in S_l(\mathcal{M})$, and $q' \in S_l(\mathcal{M}')$, $q \subseteq q'$. Then $p' \times q'$ is an extension of $p \times q$.*

PROPOSITION 3. *Let $p \in S_n(\mathcal{M})$, $q \in S_l(\mathcal{M})$, p and q definable. Then $p \times q$ is definable.*

In [9], we gave a proof of the following theorem, using the notion of φ-rank. The idea of the proof presented here was suggested by Lachlan.

THEOREM 4 (commutativity of the product of types). *Suppose T is stable, and let $\mathcal{M} \subseteq \mathcal{M}'$, $\bar{a} \in M'^n$, $\bar{b} \in \bar{M}'^l$, $p \in S_n(\mathcal{M})$, $q \in S_l(\mathcal{M})$. If $t((\bar{a}, \bar{b}), \mathcal{M}) = p \times q$, then $t((\bar{b}, \bar{a}), \mathcal{M}) = q \times p$.*

PROOF. Let $\mathcal{M}_1 \supseteq \mathcal{M}$ such that $\mathcal{M}_1$ is $\| M \|^+$-saturated. Let d and d' be schemas defining p and q; define the sequences $(\bar{a}_k ; k \in \omega)$ and $(\bar{b}_k ; k \in \omega)$ by induction on k such that:

$$- \bar{a}_k \in M_1^n \quad \text{and } \bar{a}_k \text{ realizes } \quad d\Big(\mathcal{M} \cup \bigcup_{j<k} |\bar{a}_j| \cup \bigcup_{j<k} |\bar{b}_j|\Big)$$

over $\mathcal{M} \cup \bigcup_{j<k} |\bar{a}_j| \cup \bigcup_{j<k} |\bar{b}_j|$. $-\ \bar{b}_k \in M_1^l$ and $\bar{b}_k$ realizes $d'(\mathcal{M} \cup \bigcup_{j\leqq k} |\bar{a}_j| \cup \bigcup_{j<k} |\bar{b}_j|)$ over $\mathcal{M} \cup \bigcup_{j\leqq k} |\bar{a}_j| \cup \bigcup_{j<k} |\bar{b}_j|$.

By Proposition 2, 9, if $i > j$, $\bar{a}_i$ realizes the heir of p on $\mathcal{M} \cup |\bar{b}_j|$, and therefore $(\bar{a}_i \hat{}\ \bar{b}_j)$ realizes $p \times q$ over $\mathcal{M}$. Similarly, if $j \geqq i$, $(\bar{b}_j \hat{}\ \bar{a}_i)$ realizes $q \times p$ over $\mathcal{M}$. If we suppose the theorem false, then there exists $\varphi \in L_{n+l}(M)$ such that:

$$\text{if} \quad i > j, \quad \text{then} \quad \mathcal{M}_1 \models \neg\varphi(\bar{b}_j, \bar{a}_i)$$

$$\text{if} \quad j \geqq i, \quad \text{then} \quad \mathcal{M}_1 \models \varphi(\bar{b}_j, \bar{a}_i).$$

Therefore, φ has the order property (see [16]), and T is not stable.

Remark. If we do not suppose T stable, it is possible for p and q to be definable complete types on $\mathcal{M}$, and for the conclusion of Theorem 4 to fail: this is shown by the example of ultrafilters over ω (Example 1 of Section 2).

We shall suppose, until the end of this section, that T is stable. If $\mathcal{M} \subseteq \mathcal{A}$, we shall denote by $h^n_{\mathcal{M},\mathcal{A}}$ the map from $S_n(\mathcal{M})$ into $S_n(\mathcal{A})$, which maps $p \in S_n(\mathcal{M})$ to its heir on $\mathcal{A}$. Then:

Theorem 5. *$h^n_{\mathcal{M},\mathcal{A}}$ is a continuous map.*

Proof. We have to show that if U is a clopen set of $S_n(\mathcal{A})$ then $(h^n_{\mathcal{M},\mathcal{A}})^{-1}(U)$ is an open set of $S_n(\mathcal{M})$. Suppose that

$$U = \{p\,;\, p \in S_n(\mathcal{A}) \text{ and } \varphi(\bar{x}_n, \bar{a}) \in p\} \quad \text{where} \quad \varphi \in L_{n+k} \quad \text{and} \quad \bar{a} \in A^k.$$

Let $\mathcal{M}' \supseteq \mathcal{A}$, $\mathcal{M}'$ is $(\|A\|)^+$-saturated, and $q \in S_n(\mathcal{M})$. Then the following are equivalent:

1) $\varphi(\bar{x}_n, \bar{a}) \in h^n_{\mathcal{M},\mathcal{A}}(q)$.

2) For any $\bar{b} \in M'^n$, if $t(\bar{b}, \mathcal{M} \cup \bar{a})$ is the heir of q on $\mathcal{M} \cup \bar{a}$, then $\mathcal{M}' \models \varphi(\bar{b}, \bar{a})$.

3) For any $\bar{b} \in M'^n$, if $t(\bar{b}, \mathcal{M}) = q$ and $t(\bar{a}, \mathcal{M} \cup \bar{b})$ is the heir of $t(\bar{a}, \mathcal{M})$, then $\mathcal{M}' \models \varphi(\bar{b}, \bar{a})$.

Let d be a k-schema on $\mathcal{M}$ defining $t(\bar{a}, \mathcal{M})$, and set $\varphi_1 = \varphi(\bar{v}_n, \bar{x}_k)$. Then 3) is equivalent to:

4) For any $\bar{b} \in M'^n$, if $t(\bar{b}, \mathcal{M}) = q$, then $\mathcal{M}' \models d(\varphi_1)(\bar{b})$. Therefore $(h^n_{\mathcal{M},\mathcal{A}})^{-1}(U) = \{q\,;\, q \in S_n(\mathcal{M}) \text{ and } d(\varphi_1)(\bar{x}_n) \in q\}$ and this is a clopen set of $S_n(\mathcal{M})$.

Remark 1. This theorem proves that the map $h^n_{\mathcal{M},\mathcal{A}}$ is a continuous section of $i^n_{\mathcal{A},\mathcal{M}}$. This property is characteristic of $h^n_{\mathcal{M},\mathcal{A}}$. Indeed if h' is another continuous section of $i^n_{\mathcal{A},\mathcal{M}}$, then h' and $h^n_{\mathcal{M},\mathcal{A}}$ are identical on the set $\{t(\bar{m}, \mathcal{M})\,;\, \bar{m} \in M^n\}$ which is dense in $S_n(\mathcal{M})$.

REMARK 2. Theorems 4) and 5) imply that the map from $S_n(\mathcal{M})\times S_l(\mathcal{M})$ into $S_{n+l}(\mathcal{M})$ which maps (p,q) to $p\times q$ is continuous in each of the two variables. But it is not a continuous map: suppose $n=l=1$, for the sake of simplicity.

The set $x=\{t((a,b);\mathcal{M});(a,b)\in M^2\}$ is dense in $S_2(\mathcal{M})$; on the other hand, if $(a,b)\in M^2$, then $t((a,b),\mathcal{M})=t(a,\mathcal{M})\times t(b,\mathcal{M})$. If we suppose the product continuous, then its range is a closed set and includes x. So it is an onto map; but we can easily see that if $p\in S_1(\mathcal{M})$ and p is not realized in $\mathcal{M}$, then the 2-type generated by $p\cup\{x_0=x_1\}$ is not the product of two complete 1-types on $\mathcal{M}$.

The proof of the following lemma is easy.

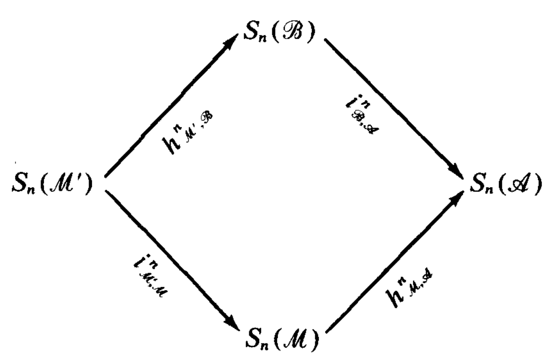
LEMMA 6. *Let $\mathcal{M}\subseteq\mathcal{A}$ and $p\in S_n(\mathcal{A})$. If for all $\mathcal{A}_1$ such that $\mathcal{M}\subseteq\mathcal{A}_1\subseteq\mathcal{A}$ and A_1-M is finite, $p\restriction\mathcal{A}_1$ is the heir of $p\restriction\mathcal{M}$, then p is the heir of $p\restriction\mathcal{M}$.*

DEFINITION 7. Let $\mathcal{M}\subseteq\mathcal{C}$, $\mathcal{M}\subseteq\mathcal{A}\subseteq\mathcal{C}$ and $\mathcal{M}\subseteq\mathcal{B}\subseteq\mathcal{C}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *independent* over $\mathcal{M}$, if for every $\bar a\in\bar A$ and $\bar b\in\bar B$, $t(\bar a,\mathcal{M}\cup\bar b)$ is the heir of $t(\bar a,\mathcal{M})$.

By Theorem 4, we see that the independence relation is symmetrical. By Lemma 6, if $\mathcal{A}$ and $\mathcal{B}$ are independent over $\mathcal{M}$, and $\bar a\in\bar A$, then $t(\bar a,\mathcal{B})$ is the heir of $t(\bar a,\mathcal{M})$.

THEOREM 8. *Let $\mathcal{M}\subseteq\mathcal{M}'\subseteq\mathcal{B}$ and $\mathcal{M}\subseteq\mathcal{A}\subseteq\mathcal{B}$, and suppose that $\mathcal{A}$ and $\mathcal{M}'$ are independent over $\mathcal{M}$. Then $h^n_{\mathcal{M},\mathcal{A}}\circ i^n_{\mathcal{M}',\mathcal{M}}=i^n_{\mathcal{B},\mathcal{A}}\circ h^n_{\mathcal{M}',\mathcal{B}}$.*

This means that the diagram:



commutes.

PROOF. All the functions which we utilize are continuous. Since $\{t(\bar m,\mathcal{M}');\bar m\in M'^n\}$ is dense in $S_n(\mathcal{M}')$, it is sufficient to prove that, for every $\bar m\in M'^n$ we have:

$$h^n_{\mathcal{M},\mathcal{A}} \circ i^n_{\mathcal{M}',\mathcal{M}}(t(\bar{m}, \mathcal{M}')) = i^n_{\mathcal{B},\mathcal{A}} \circ h^n_{\mathcal{M}',\mathcal{B}}(t(\bar{m}, \mathcal{M}')).$$

Let $\bar{m} \in M'^n$. Then $t(\bar{m}, \mathcal{M}')$ has a unique complete extension to $\mathcal{B}$, viz. its heir on $\mathcal{B}$, which is also $t(\bar{m}, \mathcal{B})$. On the other hand $i^n_{\mathcal{B},\mathcal{A}}(t(\bar{m}, \mathcal{B})) = t(\bar{m}, \mathcal{A})$. Therefore:

$$i^n_{\mathcal{B},\mathcal{A}} h^n_{\mathcal{M}',\mathcal{B}}(t(\bar{m}, \mathcal{M}')) = t(\bar{m}, \mathcal{A}).$$

We also know that $i^n_{\mathcal{M}',\mathcal{M}}(t(\bar{m}, \mathcal{M}')) = t(\bar{m}, \mathcal{M})$, and the hypothesis implies that $t(\bar{m}, \mathcal{A})$ is the heir of $t(\bar{m}, \mathcal{M})$. Therefore:

$$h^n_{\mathcal{M},\mathcal{A}} \circ i^n_{\mathcal{M}',\mathcal{M}}(t(\bar{m}, \mathcal{M}')) = t(\bar{m}, \mathcal{A}).$$

Proposition 9. *Let $\mathcal{M} \subseteq \mathcal{A}, \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{M}'$, and suppose that $\mathcal{M}'$ is $(\| A \| + \| B \|)^+$-saturated. Then there is $\mathcal{A}' \subseteq \mathcal{M}'$ which is $\mathcal{M}$-isomorphic to $\mathcal{A}$, and such that $\mathcal{A}'$ and $\mathcal{B}$ are independent over $\mathcal{M}$.*

Proof. First consider the case where $A - M$ is finite. Let $\bar{a}$ be a sequence which enumerates $A - M$, and $\bar{a}'$ be such that $\bar{a}' \in \bar{M}'$ and $t(\bar{a}', \mathcal{B})$ is the heir of $t(\bar{a}, \mathcal{M})$. By Proposition 2, 9, 5), it should be clear that $\mathcal{B}$ and $\mathcal{M} \cup \bar{a}'$ are independent over $\mathcal{M}$.

For the general case, introduce for each $a \in A$ a new constant symbol y_a, and consider:

$$E = \{\varphi(y_{a_0}, y_{a_1}, \cdots, y_{a_{n-1}});\ n \in \omega,\ (a_0, \cdots, a_{n-1}) \in A^n,\ \varphi \in L_n(\mathcal{B}) \text{ and}$$

$$\varphi(\bar{x}_n) \in h^n_{\mathcal{M},\mathcal{B}}(t(a_0, \cdots, a_{n-1}), \mathcal{M})\}.$$

By the first part of this proof, every finite subset of E can be interpreted in $\mathcal{M}'$. Therefore E can be interpreted in $\mathcal{M}'$; if $f(a)$ is the element of M' which interprets y_a (in a fixed interpretation of E) then f is an $\mathcal{M}$-monomorphism from $\mathcal{A}'$ into $\mathcal{M}'$, and for every $\bar{a} \in \bar{A}$, $t(f(\bar{a}), \mathcal{B})$ is the heir of $t(\bar{a}, \mathcal{M}) = t(f(\bar{a}), \mathcal{M})$.

4. Ranks

In this section, we shall suppose that L does not contain any constant on function symbols, and that T is stable. We denote by On^* the class of ordinals plus one element ∞, being understood that $\alpha < \infty$ for any ordinal α. We set $S^*(T) = \bigcup_{n \in \omega} \bigcup_{\mathcal{A} \in K(T)} S_n(\mathcal{A})$.

Definition. *A rank-notion* is a map R from $S^*(T)$ into On^*, satisfying the following axioms:

1/ If $\mathcal{A} \subseteq \mathcal{B}$ and $p \in S_n(\mathcal{B})$, then $R(p) \leqq R(p \restriction \mathcal{A})$.

2/ If f is an isomorphism from $\mathscr{A}$ on to $\mathscr{A}'$, and if $p \in S_n(\mathscr{A}')$ then $R(p) = R(\hat{f}(p))$.

3/ If $\mathscr{A} \subseteq \mathscr{B}$ and $p \in S_n(\mathscr{A})$, then there exist $p_1 \in S_n(\mathscr{B})$ such that $p \subseteq p_1$, and $R(p) = R(p_1)$.

4/ For $\mathscr{A}$ and $p \in S_n(\mathscr{A})$, there is a cardinal λ such that, for all $\mathscr{B} \supseteq \mathscr{A}$, if $R(p) < \infty$,

$$\|\{p_1; p_1 \in S_n(\mathscr{B}), p \subseteq p_1 \quad \text{and} \quad R(p) = R(p_1)\}\| \leqq \lambda.$$

5/ If $p \in S_n(\mathscr{A})$, there exists $\mathscr{A}_0 \subseteq \mathscr{A}$ with A_0 finite such that $R(p) = R(p \restriction \mathscr{A}_0)$.

Axioms 1/ and 2/ are equivalent to the following single axiom:

1'/ If f is a monomorphism from $\mathscr{A}$ into $\mathscr{A}'$, and $p \in S_n(\mathscr{A}')$, then $R(p) \leqq R(\hat{f}(p))$.

Most of the ordinal-valued ranks which have been defined satisfy these conditions. Morley, in [12], defined a rank-notion which we shall denote by R_0. (In fact, he defines R_0 only for 1-types, but there is no difficulty in generalizing his definition.)[(†)]

It should be noted that if T is superstable, then there exists a rank-notion R such that, for every $p \in S^*(T)$, $R(p) < \infty$. Take for example Deg as defined in [15], or rank as defined in [17]. The converse is true (cf. [15] and the remarks at the end of this section).

In the following, R will always denote a rank-notion; $R(\bar{a}, \mathscr{A})$ will be $R(t(\bar{a}, \mathscr{A}))$.

Definition 2. Let R and R' be two rank-notions. We say that R and R' are *equivalent* if for any $\mathscr{A} \subseteq \mathscr{B}$, $n \in \omega$ and $p \in S_n(\mathscr{B})$ such that $R(p) < \infty$ and $R'(p) < \infty$, $R(p) = R(p \restriction \mathscr{A})$ if and only if $R'(p) = R'(p \restriction \mathscr{A})$.

Proposition 3. *Suppose $p \in S_n(\mathscr{A})$, and $R(p) < \infty$. Then there exists $\mathscr{M} \supseteq \mathscr{A}$, such that, for any $\mathscr{B} \supseteq \mathscr{M}$ and p_1, p_2, extensions of p in $S_n(\mathscr{B})$ such that $R(p_1) = R(p_2) = R(p)$, if $p_1 \neq p_2$ then $p_1 \restriction \mathscr{M} \neq p_2 \restriction \mathscr{M}$.*

Proof. Let λ be the cardinal given by axiom 4/ in Definition 1, and $\mathscr{M} \supseteq \mathscr{A}$; $\mathscr{M}$ is $(\|A\| + \lambda + \|T\|)^+$-saturated. We shall see that $\mathscr{M}$ fulfills the required condition.

For any $q_1, q_2 \in S_n(\mathscr{M})$, $q_1 \neq q_2$, let $\mathscr{C}(q_1, q_2)$ be a finite substructure of $\mathscr{M}$ such

[(†)] For superstable theories, axiom 5 follows from axioms 1–4. Furthermore the theorems of this section can be extended to rank-notion for stable theories satisfying only axioms 1–4. This more general result makes use of Shelah's notion of forking.

that $q_1 \restriction \mathscr{C}(q_1, q_2) \neq q_2 \restriction \mathscr{C}(q_1, q_2)$ and set $\mathscr{C} = \mathscr{A} \cup \bigcup \{\mathscr{C}(q_1, q_2);$ $q_1, q_2 \in S_n(\mathscr{M}), q_1 \neq q_2, p \subseteq q_1, p \in q_2$ and $R(q_1) = R(q_2) = R(p)\}$. We see that $\| C \| \leqq \lambda + \| A \|$, and if q_1, and q_2 are two distinct extensions of p in $S_n(\mathscr{M})$ of the same rank, then $q_1 \restriction \mathscr{C} \neq q_2 \restriction \mathscr{C}$.

Suppose now that

$$\mathscr{B} \supseteq \mathscr{M}, p_1, p_2 \in S_n(\mathscr{B}), p \subseteq p_1, p \subseteq p_2, p_1 \neq p_2, R(p_1) = R(p_2) = R(p).$$

We may suppose that $B - M$ is finite, and let $\bar{b}$ be a finite sequence which enumerates this set. There exist $\bar{b}_1 \in \bar{M}$ such that $t(\bar{b}_1, \mathscr{C}) = t(\bar{b}, \mathscr{C})$ and a $\mathscr{C}$-isomorphism f from $\mathscr{C}_1 \cup \bar{b}$ onto $\mathscr{C} \cup \bar{b}_2$. Therefore $\hat{f}(p_1 \restriction \mathscr{C} \cup \bar{b})$ and $\hat{f}(p_2 \restriction \mathscr{C} \cup \bar{b})$ are distinct extensions of p in $S_n(\mathscr{C} \cup \bar{b}_1)$ of the same rank. So $\hat{f}(p_1 \restriction \mathscr{C} \cup \bar{b}) \restriction \mathscr{C} \hat{f}(p_2 \restriction \mathscr{C} \cup \bar{b}) \restriction \mathscr{C}$ but $\hat{f}(p_1 \restriction \mathscr{C} \cup \bar{b}) \restriction \mathscr{C} = \hat{f}(p_1 \restriction \mathscr{C}) = p_1 \restriction \mathscr{C}$ since f is a $\mathscr{C}$-isomorphism, and similarly $\hat{f}(p_2 \restriction \mathscr{C} \cup \bar{b}) = p_2 \restriction \mathscr{C}$. Therefore $p_1 \restriction \mathscr{C} \neq p_2 \restriction \mathscr{C}$ and $p_1 \restriction \mathscr{M} \neq p_2 \restriction \mathscr{M}$.

THEOREM 4. *Let $p \in S_n(\mathscr{M})$, $\mathscr{A} \supseteq \mathscr{M}$ and p' the heir of p on $\mathscr{A}$. Then*:

1/ $R(p') = R(p)$.

2/ *If $R(p) < \infty$, then p' is the unique extension of p in $S_n(\mathscr{A})$ with $R(p') = R(p)$.*

PROOF. We shall prove the theorem by induction: suppose it is true for every $\mathscr{M}, \mathscr{A}, p$, provided that $R(p) < \alpha$ (with $\alpha \in On^*$). Clearly, in order to prove that 1/ is true when $R(p) = \alpha$, we may suppose that $\mathscr{A}$ is a model of T. So set

$\beta = \inf\{R(p');$ $\mathscr{M}$ is a model of T, $\mathscr{M}_1 \supseteq \mathscr{M}$, $p \in S_n(\mathscr{M})$, $R(p) = \alpha$ and p' is the heir of p on $\mathscr{M}_1\}$

and suppose for contradiction that $\beta < \alpha$. There exist $\mathscr{M}$, $\mathscr{M}_1$, p, p' as required such that $R(p') = \beta$. Let $\mathscr{M}'$ be a $\| M_1 \|^+$-saturated model of T containing $\mathscr{M}_1$. By Proposition 3,9, there exists $\mathscr{M}_2$ such that $\mathscr{M} \subseteq \mathscr{M}_2 \subseteq \mathscr{M}'$, $\mathscr{M}_2$ is $\mathscr{M}$-isomorphic to $\mathscr{M}$, and $\mathscr{M}_1$ and $\mathscr{M}_2$ are independent over $\mathscr{M}$. Let q be an extension of p in $S_n(\mathscr{M}_2)$ such that $R(q) = R(p)$. By isomorphism, q is not the heir of p.

Now by Theorem 3,8, if q' is the heir of q on $\mathscr{M}'$, q' is an extension of p'. By the definition of β, $R(q') \geqq \beta$, and since $R(p') = \beta$, we see that $R(q') = R(p') = \beta$. But $\beta < \infty$, and by the induction hypothesis, q' is the heir of p', so it is the heir of p. Therefore, q, which is its restriction to $\mathscr{M}_2$, is the heir of p, which is impossible.

Let us turn to part 2/. Suppose $p \in S_n(\mathscr{M})$, $R(p) = \alpha$, and q is an extension of p in $S_n(\mathscr{A})$ with $R(q) = \alpha$. By Proposition 3, there is $\mathscr{M}_1$, such that, for every $\mathscr{B} \supseteq \mathscr{M}_1$, and p_1, p_2 extensions of p in $S_n(\mathscr{B})$ such that $R(p_1) = R(p_2) = R(p)$, if $p_1 \neq p_2$, then $p_1 \restriction \mathscr{M} \neq p_2 \restriction \mathscr{M}$. This implies that if q is an extension of p in $S_n(\mathscr{B})$

such that $R(q)=R(p)$, q has the unique extension in $S_n(\mathcal{B})$ of the same rank, and from the first part of the proof, this extension has to be the heir of q on $\mathcal{B}$.

Now, if $\mathcal{M}'$ is a model containing $\mathcal{A}$ and $(\|A\|+\|M_1\|)^+$-saturated, there exists $\mathcal{M}_2 \subseteq \mathcal{M}'$ which is $\mathcal{M}$-isomorphic to $\mathcal{M}_1$, which has therefore the same properties, and such that $\mathcal{A}$ and $\mathcal{M}_2$ are independent over $\mathcal{M}$. Let r be an extension of q in $S_n(\mathcal{M}')$ such that $R(r)=R(q)=R(p)$, so that r is the heir of $r \upharpoonright \mathcal{M}_2$. By Theorem 3, 8, this implies that q is the heir of p.

Corollary 5. *If $p \in S_n(\mathcal{M})$ and $R_0(p)<\infty$, then the degree of p is* 1.

This was first proved by Lachlan.

Proposition 6. *Let R and R' be rank-notions, and $\bar{b}$, $\bar{c} \in \bar{M}$, $\mathcal{A} \subseteq \mathcal{M}$, and suppose that $R(\bar{b}, \mathcal{A} \cup \bar{c}) = R(\bar{b}, \mathcal{A}) < \infty$. Then $R'(\bar{c}, \mathcal{A} \cup \bar{b}) = R'(\bar{c}, \mathcal{A})$.*

Proof. We first assume that $\mathcal{A}$ is a model of T; then by Theorem 4, $t(\bar{b}, \mathcal{A} \cup \bar{c})$ is the heir of $t(\bar{b}, \mathcal{A})$ and by Theorem 3, 4, $t((\bar{c}, \mathcal{A} \cup \bar{b})$ is the heir of $t(\bar{c}, \mathcal{A} \cup \bar{b})$. Therefore $R'(\bar{c}, \mathcal{A} \cup \bar{b}) = R'(\bar{c}, \mathcal{A})$.

To prove the general case, suppose $\mathcal{M}'$ is an elementary extension of $\mathcal{M}$ which is $\|M\|^+$-saturated, and construct first $\bar{c}_1 \in \bar{M}'$, next $\bar{b} \in \bar{M}'$, such that:

$$t(\bar{c}, \mathcal{A}) = t(\bar{c}_1, \mathcal{A}) \text{ and } R'(\bar{c}_1, \mathcal{A}) = R'(c_1, \mathcal{M}) = R'(\bar{c}, \mathcal{A})$$

$$t(\bar{b} \,^\wedge \bar{c}, \mathcal{A}) = t(\bar{b}_1 \,^\wedge \bar{c}_1, \mathcal{A}) \text{ and } R(\bar{b}_1, \mathcal{M} \cup \bar{c}_1) = R(\bar{b}_1, \mathcal{A} \cup \bar{c}_1) = R(\bar{b}, \mathcal{A} \cup \bar{c}).$$

It is clear that $t(\bar{b}_1, \mathcal{A}) = t(\bar{b}, \mathcal{A})$, $R(b_1, \mathcal{A}) = R(b, \mathcal{A})$ and $R'(\bar{c}, \mathcal{A} \cup \bar{b}) = R'(\bar{c}_1, \mathcal{A} \cup \bar{b}_1)$.

Now suppose that $R(\bar{b}, \mathcal{A} \cup \bar{c}) = R(\bar{b}, \mathcal{A}) < \infty$. Then $R(\bar{b}_1, \mathcal{A} \cup \bar{c}_1) = R(\bar{b}_1, \mathcal{A}) = R(\bar{b}_1, \mathcal{M} \cup \bar{c}_1) < \infty$, and we may deduce, by axiom 1/

$$R(\bar{b}_1, \mathcal{M}) = R(\bar{b}_1, \mathcal{M} \cup \bar{c}).$$

By the first part of the proof, we get

$$R'(\bar{c}_1, \mathcal{M} \cup \bar{b}_1) = R'(\bar{c}_1, \mathcal{M}) = R'(\bar{c}_1, \mathcal{A})$$

and, again by axiom 1/

$$R'(\bar{c}_1, \mathcal{A}) = R'(\bar{c}_1, \mathcal{A} \cup \bar{b}_1) = R'(\bar{c}, \mathcal{A}) = R'(\bar{c}, \mathcal{A} \cup \bar{b}).$$

By taking $R' = R$, we get:

Theorem 7 (reciprocity principle). *Let $\bar{b}, \bar{c} \in \bar{M}$, $\mathcal{A} \subseteq \mathcal{M}$, and suppose $R(\bar{b}, \mathcal{A} \cup \bar{c}) < \infty$ and $R(\bar{c}, \mathcal{A} \cup \bar{b}) < \infty$. Then*

$$R(\bar{b}, \mathcal{A} \cup \bar{c}) = R(\bar{b}, \mathcal{A}) \text{ if } R(\bar{c}, \mathcal{A} \cup \bar{b}) = R(\bar{c}, \mathcal{A}).$$

REMARK. This principle generalizes the exchange principle: Let $\mathscr{A} \subseteq \mathscr{M}$ such that for every $a \in A$, $R(a) = 1$, or even more generally such that the rank of any complete extension of $t(a)$ is 0 or equal to $R(a)$. Then, for all $\mathscr{B} \subseteq \mathscr{M}$, $a \in A$, and $c \in M$, if $t(c, \mathscr{B})$ is not algebraic, but $t(c, \mathscr{B} \cup a)$ is, then $t(a, \mathscr{B} \cup c)$ is algebraic.

PROPOSITION 8. *Let $\mathscr{A} \subseteq \mathscr{M}$, $\bar{b}, \bar{c} \in \bar{M}$, and suppose $R_0(\bar{b}, \mathscr{A} \cup \bar{c}) = R_0(\bar{b}, \mathscr{A}) < \infty$ and $R_0(\bar{c}, \mathscr{A} \cup \bar{b}) < \infty$; then $d(\bar{b}, \mathscr{A} \cup \bar{c}) = d(\bar{b}, \mathscr{A})$ if and only if $d(\bar{c}, \mathscr{A} \cup \bar{b}) = d(\bar{c}, \mathscr{A})$.*

Here $d(\bar{b}, \mathscr{A})$ denotes the degree of $t(\bar{b}, \mathscr{A})$ as defined in [12].

PROOF. Given the hypothesis, the following are equivalent:

$d(\bar{c}, \mathscr{A}) = d(\bar{c}, \mathscr{A} \cup \bar{b})$.

$t(\bar{c}, \mathscr{A})$ has a unique complete extension of same rank over $\mathscr{A} \cup \bar{b}$.

For any $\mathscr{M}' \supseteq \mathscr{M}, \overline{b'}, \overline{c'} \in \overline{M'}$ such that $t(\bar{b}, \mathscr{A}) = t(\overline{b'}, \mathscr{A}), t(\bar{c}, \mathscr{A}) = t(c', \mathscr{A})$ and $R_0(\overline{b'}, \mathscr{A} \cup \overline{c'}) = R_0(b', \mathscr{A})$, we have $t(\overline{b'} \hat{} \overline{c'}, \mathscr{A}) = t(\bar{b} \hat{} \bar{c}, \mathscr{A})$.

This last condition is symmetric in $\bar{b}$ and $\bar{c}$, and the proposition is proved.

EXAMPLE. Dickmann [6, p. 179] asks the following: Let $\mathscr{A} \subseteq \mathscr{M}$ such that, for any $a \in A$, $R_0(a) = 1$, and $d(a) = 1$, and suppose that $\mathscr{A}$ is algebraically independent. Let $b \in M$, an algebraic point (that is $R_0(b) = 0$).

Is it always true that $d(b) = d(b, \mathscr{A})$?

Answer: Yes.

Otherwise there would exist $\mathscr{A}_0 \subseteq \mathscr{A}$, A_0 finite, and $a \in A$ such that $d(b, \mathscr{A}_0 \cup a) < d(b, \mathscr{A}_0)$. But

$$R_0(b, \mathscr{A}_0 \cup a) = R_0(b, \mathscr{A}) = 0.$$

Therefore $R_0(a, \mathscr{A}_0 \cup b) = R_0(a, \mathscr{A}_0) = 1$, and $d(a, \mathscr{A}_0 \cup b) = d(a, \mathscr{A}_0) = 1$ and this is impossible.

From now on and until the end of the Corollary 17, we shall suppose that T is superstable; R_1 will denote a rank-notion such that, for any $p \in S^*(T)$, $R_1(p) < \infty$.

THEOREM 9. *Any two rank-notions are equivalent.*

PROOF. Let R and R' be rank-notions, $\mathscr{A} \subseteq \mathscr{B}$, $p \in S_n(\mathscr{B})$, and suppose that $R(p) = R(p \upharpoonright \mathscr{A}) < \infty$. We have to prove that $R'(p) = R'(p \upharpoonright \mathscr{A})$.

Let $\mathscr{M} \supseteq \mathscr{B}$, and $\bar{c} \in M^n$ be such that $t(\bar{c}, \mathscr{B}) = p$; then for any $\bar{b} \in \bar{B}$, we have

$$R(\bar{c}, \mathscr{A}) = R(\bar{c}, \mathscr{A} \cup \bar{b}) < \infty.$$

By Proposition 6, applied to R, R_1,

$$R_1(\bar{b}, \mathscr{A} \cup \bar{c}) = R_1(\bar{b}, \mathscr{A})$$

and, since $R_1(\bar{b}, \mathscr{A} \cup \bar{c}) < \infty$,

$$R'(\bar{c}, \mathscr{A} \cup \bar{c}) = R'(\bar{c}, \mathscr{A}).$$

And this, together with axiom 5, easily implies

$$R'(\bar{c}, \mathscr{B}) = R'(\bar{c}, \mathscr{A}) = R'(p) = R'(p \restriction \mathscr{A}).$$

DEFINITION 10. Let $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{M}$ and $\mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{M}$. We say that $\mathscr{B}$ and $\mathscr{C}$ are *independent over* $\mathscr{A}$, if for any $\bar{b} \in \bar{B}$ and $\bar{c} \in \bar{C}$ we have

$$R_1(\bar{b}, \mathscr{A}) = R_1(\bar{b}, \mathscr{A} \cup \bar{c}).$$

This definition requires some comments. First of all, it is not contradictory with Definition 3, 7, as follows from Theorem 4. Second, the notion of independence does not depend upon the particular rank-notion R_1 chosen (provided that, for any $p \in S^*(T)$, $R_1(p) < \infty$): this is a consequence of Theorem 9. Finally the independence relation is symmetric (Theorem 6), and by axiom 5, if $\mathscr{B}$ and $\mathscr{C}$ are independent over $\mathscr{A}$, then for any $\bar{b} \in \bar{B}$,

$$R_1(\bar{b}, \mathscr{A}) = R_1(\bar{b}, \mathscr{A} \cup \mathscr{C}).$$

PROPOSITION 11. *Let $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{M}$ and $\mathscr{A} \subseteq \mathscr{C}$, and suppose that $\mathscr{M}$ is $(\| B \| + \| C \|)^+$-saturated. Then there exists $\mathscr{C}'$ such that $\mathscr{A} \subseteq \mathscr{C}' \subseteq \mathscr{M}$, $\mathscr{C}'$ is $\mathscr{A}$-isomorphic to $\mathscr{C}$, and $\mathscr{C}'$ and $\mathscr{B}$ are independent over $\mathscr{A}$.*

PROOF. Consider the set

$$K = \{(\mathscr{C}_1, \mathscr{C}'_1, f);\ \mathscr{A} \subseteq \mathscr{C}_1 \subseteq \mathscr{C}, \mathscr{A} \subseteq \mathscr{C}'_1 \subseteq \mathscr{M}, \mathscr{B} \text{ and } \mathscr{C}'_1$$
$$\text{are independent, and } f \text{ is an } \mathscr{A}\text{-isomorphism from } \mathscr{C}'_1 \text{, onto } \mathscr{C}_1\}$$

and define the order on K by setting

$$(\mathscr{C}_1, \mathscr{C}'_1, f_1) < (\mathscr{C}_2, \mathscr{C}'_2, f_2) \text{ if and only if } \mathscr{C}_1 \subseteq \mathscr{C}_2, \mathscr{C}'_1 \subseteq \mathscr{C}',$$

and f_2 extends f_1.

This set is not emply (since $(\mathscr{A}, \mathscr{A}, e_{\mathscr{A},\mathscr{A}}) \in K$), and it is clear that any linearly ordered subset of K is bounded in K. So we may apply Zorn's lemma: let $(\mathscr{C}_0, \mathscr{C}'_0, f_0)$ be maximal in K.

Let $c \in C$. Then there is $c' \in M'$ such that

$$t(c', \mathscr{C}'_0) = \hat{f}'_1(t(c, \mathscr{C}_0))$$

$$\text{and } R_1(c', \mathscr{C}'_0) = R_1(c', \mathscr{C}'_0 \cup \mathscr{B}).$$

From this, for any $\bar{b} \in \bar{B}, R_1(\bar{b}, \mathscr{C}'_0) = R_1(\bar{b}, \mathscr{C}'_0 \cup c')$. But we know that $R_1(\bar{b}, \mathscr{C}'_0) = R_1(\bar{b}, \mathscr{A})$; therefore $\mathscr{B}$ and $\mathscr{C}'_0 \cup c'$ are independent, and by the maximality of $(\mathscr{C}_1, \mathscr{C}'_1, f_1)$, $c \in C_0$ and $C_0 = C$.

THEOREM 12. *Let $\mathscr{A} \subseteq \mathscr{M}_1, p \in S_n(\mathscr{A})$, $\mathscr{M}_1 \subseteq \mathscr{M}$, and suppose that $\| M \| \geqq \| M_1 \|^+$ and $\mathscr{M}$ is saturated. If p_1 and p_2 are two extensions of p in $S_n(\mathscr{M})$ such that $R_1(p_1) = R_1(p_2)$, then there is an $\mathscr{A}$-automorphism f of $\mathscr{M}$ such that $\hat{f}^n(p_2) = p_1$.*

PROOF. Let $p'_1 = p_1 \restriction \mathscr{M}_1$, and $p'_2 = p_2 \restriction \mathscr{M}_1$ and let $\bar{a}$ and $\bar{a}_2$ be elements of $\bar{M}$ realizing p'_1 and p'_2 over $\mathscr{M}_1$ respectively. Since $t(\bar{a}, \mathscr{A}) = t(\bar{a}_2, \mathscr{A}) = p$, there is an $\mathscr{A}$-automorphism h of $\mathscr{M}$ such that $h(\bar{a}) = \bar{a}_2$. Let $M'_2 = h^{-1}(M_1)$ and call h' the restriction of h to $\mathscr{M}'_2$, so that h' is an isomorphism from $\mathscr{M}'_2$ onto $\mathscr{M}_1$. We have

$$t(\bar{a}, \mathscr{M}'_2) = \hat{h}'^n(t(\bar{a}_2, \mathscr{M}_1)) = \hat{h}'^n(p'_2).$$

Now, by Proposition 11, there is $\mathscr{M}_2 \subseteq \mathscr{M}$, $\mathscr{A} \subseteq \mathscr{M}_2$ and g an $(\mathscr{A} \cup \bar{a})$-isomorphism from $\mathscr{M}_2 \cup \bar{a}$ onto $\mathscr{M}'_2 \cup \bar{a}$, such that $\mathscr{M}_2 \cup \bar{a}$ and $\mathscr{M}_1 \cup \bar{a}$ are independent over $\mathscr{A} \cup \bar{a}$. Let g' be the restriction of g to $\mathscr{M}_2$, so that g' is an isomorphism from $\mathscr{M}_2$ onto $\mathscr{M}'_2$, and $h' \circ g'$ is an isomorphism from $\mathscr{M}_2$ onto $\mathscr{M}_1$. We have

$$t(\bar{a}, \mathscr{M}_2) = \hat{g}'^n(t(\bar{a}, \mathscr{M}'_2)) = \widehat{(h' \circ g')}^n(p'_2).$$

For any $\bar{m} \in \bar{M}_1$, we have

$$R_1(\bar{m}, \mathscr{A} \cup \bar{a}) = R_1(\bar{m}, \mathscr{M}_2 \cup \bar{a})$$

since $\mathscr{M}_2 \cup \bar{a}$ and $\mathscr{M}_1 \cup \bar{a}$ are independent over $\mathscr{A} \cup \bar{a}$. On the other hand, our hypothesis is:

$$R_1(\bar{a}, \mathscr{A}) = R_1(\bar{a}, \mathscr{M}_1) = R_1(\bar{a}, \mathscr{A} \cup \bar{m})$$

and by the reciprocity principle

$$R_1(\bar{m}, \mathscr{A}) = R_1(\bar{m}, \mathscr{A} \cup \bar{a}).$$

Therefore

$$R_1(\bar{m}, \mathscr{A}) = R_1(\bar{m}, \mathscr{M}_2 \cup \bar{a}) = R_1(\bar{m}, \mathscr{M}_2)$$

and

$$R_1(\bar{a}, \mathscr{M}_2) = R_1(\bar{a}, \mathscr{M}_2 \cup \bar{m}).$$

Recall that this is true for any $\bar{m} \in \bar{M}_1$. So

$$R_1(\bar{a}, \mathcal{M}_2) = R_1(\bar{a}, \mathcal{M}_1 \cup \mathcal{M}_2)$$

and

$$R_1(\bar{a}, \mathcal{M}_2) = R_1(\bar{a}, \mathcal{A}), \text{ since } R_1(\bar{a}, \mathcal{M}_2) = R_1(p_2').$$

We see that $t(\bar{a}, \mathcal{M}_1 \cup \mathcal{M}_2)$ and $p_1 \restriction \mathcal{M}_1 \cup \mathcal{M}_2$ are two extensions of p_1', of equal rank. So by Theorem 4, $t(\bar{a}, \mathcal{M}_1 \cup \mathcal{M}_2) = p_1 \restriction \mathcal{M}_1 \cup \mathcal{M}_2$ and $t(\bar{a}, \mathcal{M}_2) \subseteq p_1$.

Now let f be an $\mathcal{A}$-automorphism of $\mathcal{M}$ which extends $h' \circ g'$. Then $\hat{f}^n(p_2)$ is an extension of $(h' \hat{\circ} g')^n (p_2') = t(\bar{a}, \mathcal{M}_2)$ and we have seen that the same holds for p_1. But they still have the same rank as p, and therefore $\hat{f}^n(p_2) = p_1$.

PROPOSITION 13. *Let $\mathcal{A} \subseteq \mathcal{B}$, and $p \in S_n(\mathcal{B})$. The three following conditions are equivalent:*

1/ $R_1(p) = R_1(p \restriction \mathcal{A})$.

2/ *For any $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{B} \subseteq \mathcal{M}'$, there is an extension q of p in $S_n(\mathcal{M}')$ such that q is the heir of $q \restriction \mathcal{M}$.*

3/ *There exist $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{M}'$, $\mathcal{B} \subseteq \mathcal{M}'$ and $\mathcal{B}$ and $\mathcal{M}$ are independent over $\mathcal{A}$, and an extension q of p in $S_n(\mathcal{M}')$ such that q is the heir of $q \restriction \mathcal{M}$.*

PROOF.

1/$\to$2/. Let q be an extension of p in $S_n(\mathcal{M}')$ such that $R_1(q) = R_1(p) = R_1(p \restriction \mathcal{A})$. By axiom 1, $R_1(q) = R_1(q \restriction \mathcal{M})$, and q is the heir of $q \restriction \mathcal{M}$.

2/$\to$3/ follows immediately from Proposition 11.

3/$\to$1/. Let $\mathcal{M}_1 \supseteq \mathcal{M}'$, and $\bar{c} \in \bar{M}_1$ realizes q over $\mathcal{M}'$. The hypothesis implies

$$R_1(\bar{c}, \mathcal{M}') = R_1(\bar{c}, \mathcal{M}), \quad \text{and for any} \quad \bar{b} \in \bar{B}, \quad R_1(\bar{c}, \mathcal{M} \cup \bar{b}) = R_1(\bar{c}, \mathcal{M})$$

and

$$R_1(\bar{b}, \mathcal{M} \cup \bar{c}) = R_1(\bar{b}, \mathcal{M}) = R_1(\bar{b}, \mathcal{A}) = R_1(\bar{b}, \mathcal{A} \cup \bar{c});$$

consequently,

$$R_1(\bar{c}, \mathcal{B}) = R_1(\bar{c}, \mathcal{A}) \quad \text{and} \quad R_1(p) = R_1(p \restriction \mathcal{A}).$$

REMARK. For a type and one of its restrictions, the relation "to have the same rank" does not depend on the rank-notion considered. It seems normal, therefore that this relation be expressible without any mention of any rank-notion: this is precisely the content of condition 2/.

COROLLARY 14. *For $\mathscr{A} \subseteq \mathscr{B}$, and $n \in \omega$, let*

$$F^n_{\mathscr{B},\mathscr{A}} = \{p\,;\, p \in S_n(\mathscr{B}) \quad \text{and} \quad R_1(p) = R_1(p \upharpoonright \mathscr{A})\}.$$

Then $F^n_{\mathscr{B},\mathscr{A}}$ is a closed set.

PROOF. Let $\mathscr{M}, \mathscr{M}'$ be such that $\mathscr{A} \subseteq \mathscr{M} \subseteq \mathscr{M}'$, $\mathscr{B} \subseteq \mathscr{M}'$ and $\mathscr{B}$ and $\mathscr{M}$ are independent over $\mathscr{A}$. Then, by the preceding proposition

$$F^n_{\mathscr{B},\mathscr{A}} = h^n_{\mathscr{M},\mathscr{M}'} \circ i^n_{\mathscr{M},\mathscr{A}}(S^n(\mathscr{M}))$$

and we know that the image of a compact Hausdorff space under a continuous map is a closed set.

THEOREM 15. *Suppose $\mathscr{A} \subseteq \mathscr{B}$ and let $j^n_{\mathscr{B},\mathscr{A}}$ be the restriction of $i^n_{\mathscr{B},\mathscr{A}}$ to $F^n_{\mathscr{B},\mathscr{A}}$; then $j^n_{\mathscr{B},\mathscr{A}}$ is an open map. Observe also that by axiom 3/, $j^n_{\mathscr{B},\mathscr{A}}$ is an onto map.*

PROOF. For notational convenience, set $F = F^n_{\mathscr{B},\mathscr{A}}$, and $j = j^n_{\mathscr{B},\mathscr{A}}$. Let U be an open set of F, and $V = j(U)$.

1/ We first prove that V is an open set of $S_n(\mathscr{A})$ when $\mathscr{B}$ is a saturated model of T, with $\| B \| > \| A \| + \| T \|$. Let $U^* = \cup \{\hat{f}^n(U);\ f$ is an $\mathscr{A}$-automorphism of $\mathscr{B}\}$. Then U^* is an open set of F, and $j(U^*) = j(U) = V$. Let $G = F - U^*$; G is a closed set of F, and since j is onto $S_n(\mathscr{A}) - V \subseteq j(G)$. On the other hand, $j(G)$ is a closed set; so we will be done if we prove that $S_n(\mathscr{A}) - V = j(G)$ or, equivalently, $j(G) \cap V = \varnothing$. To reach a contradiction, suppose $q \in j(G) \cap V$; there is $q_1 \in G$ such that $j(q_1) = q$, and $q_2 \in U$ such that $j(q_2) = q$. By Theorem 12, there is an $\mathscr{A}$-automorphism f of $\mathscr{B}$ such that $q_1 = \hat{f}^n(q_2)$, and $q_1 \in U^*$, which is impossible.

2/ Consider now the general case. Let $\mathscr{M}$ be a saturated model which includes $\mathscr{B}$ of cardinality greater than $\| A \| + \| T \|$. There is U_1, an open set in $S_n(\mathscr{B})$, such that $U = U_1 \cap F$. Let

$$U' = (i^n_{\mathscr{M},\mathscr{B}})^{-1}(U_1) \cap F^n_{\mathscr{M},\mathscr{A}}.$$

Then U' is an open set in $F^n_{\mathscr{M},\mathscr{A}}$ and it suffices to prove that $V = j^n_{\mathscr{M},\mathscr{A}}(U)$.

If $q \in U'$, $j^n_{\mathscr{M},\mathscr{B}}(q) \in U_1$, and $R_1(j^n_{\mathscr{M},\mathscr{B}}(q)) = R_1(q \upharpoonright \mathscr{A})$ so $j^n_{\mathscr{M},\mathscr{B}}(q) \in U$ and $j^n_{\mathscr{M},\mathscr{A}}(q) = j(j^n_{\mathscr{M},\mathscr{B}}(q)) \in V$. If $p \in V$, there is $p_1 \in U$ such that $j(p_1) = p$. Let q be an extension of p_1 in $S_n(\mathscr{M})$, such that $R_1(q) = R_1(p_1) = R_1(p)$. Then $q \in U'$ and $j^n_{\mathscr{M},\mathscr{A}}(q) = p$.

COROLLARY 16. *Let $\mathscr{A} \subseteq \mathscr{B}$, $p \in S_n(\mathscr{B})$, and suppose that $R_1(p) = R_1(p \upharpoonright \mathscr{A})$ and $p \upharpoonright \mathscr{A}$ is not isolated. Then p is not isolated.*

PROOF. If p were isolated, it would be isolated in $F^n_{\mathscr{B},\mathscr{A}}$, and $p \restriction \mathscr{A}$ would also be isolated in $S_n(\mathscr{A})$.

COROLLARY 17. *Suppose T denumerable, $\mathscr{A} \subseteq \mathscr{M}$, A finite, and $\mathscr{M}$ prime over $\mathscr{A}$. Let $\mathscr{M}' \subseteq \mathscr{M}$. Then there is $\mathscr{B}_0 \subseteq \mathscr{M}'$, B_0 finite, such that $\mathscr{M}'$ is prime over $\mathscr{B}_0$.*

PROOF. Let $\bar{a}$ be a sequence which enumerates A, and consider

$$\alpha = \inf\{R_1(\bar{a}, \mathscr{B});\ \mathscr{B} \subseteq \mathscr{M}', B \quad \text{is finite}\}.$$

There is $\mathscr{B}_0 \subseteq \mathscr{M}'$, B_0 finite, such that $\alpha = R_1(\bar{a}, \mathscr{B}_0)$. Let $\bar{b}_0$ be a sequence which enumerates B_0, and $\bar{c} \in \bar{M}'$.

We see that $R_1(\bar{a}, \mathscr{B}_0 \cup \bar{c}) = R_1(\bar{a}, \mathscr{B}_0)$, and therefore $R_1(\bar{c}, \mathscr{B}_0 \cup \bar{a}) = R_1(\bar{c}, \mathscr{B}_0)$. But $t(\bar{c}, \mathscr{B}_0 \cup \bar{a})$ is isolated (because $t(\bar{c} \frown \bar{b}_0, \mathscr{A})$ is), and, by Corollary 16, $t(\bar{c}, \mathscr{B}_0)$ is isolated.

We shall conclude this chapter with some remarks. We assume now that T is stable.

1) The cardinal λ in axiom 4/ can be taken to be $2^{\|T\|}$. Let $p \in S_n(\mathscr{A})$, and $\mathscr{B} \supseteq \mathscr{A}$ and suppose $R(p) < \infty$. There is $\mathscr{A}_0 \subseteq \mathscr{A}$, A_0 finite, such that $R(p) = R(p \restriction \mathscr{A}_0)$, and let $\mathscr{M}, \mathscr{M}'$ be such that $B \subseteq \mathscr{M}'$, $\mathscr{A}_0 \subseteq \mathscr{M} \subseteq \mathscr{M}'$, $\|M\| = \|T\|$. Let μ be a cardinal, and $\{p_i; i < \mu\}$ a set of extensions of p in $S_n(\mathscr{B})$ such that, for any $i < j < \mu$, $p_i \neq p_j$ and $R(p_i) = R(p)$.

For every $i < \mu$, there is an extension q_i of p_i in $S_n(\mathscr{M})$ such that $R(q_i) = R(p_i) = R(p) = R(p \restriction \mathscr{A}_0)$, and by axiom 1/, $R(q_i) = R(q_i \restriction \mathscr{M})$. Therefore, by Theorem 4, for any $i < j < \mu$, $q_i \restriction \mathscr{M} \neq q_j \restriction \mathscr{M}$ and $\mu \leqq \|S_n(\mathscr{M})\| \leqq 2^{\|T\|}$.

2) Suppose now that there exists a rank-notion R such that, for any $p \in S^*(T)$, $R(p) < \infty$. For any $\mathscr{A} \in K(T)$, we have

$$S_1(\mathscr{A}) = \cup\{F'_{\mathscr{A},\mathscr{A}_0};\ \mathscr{A}_0 \subseteq \mathscr{A},\ A_0 \ \text{finite}\},$$

where $F'_{\mathscr{A},\mathscr{A}_0} = \{p; p \in S_1(\mathscr{A})$ and $R(p) = R(p \restriction \mathscr{A}_0\}$. By remark 1/, $\|F'_{\mathscr{A},\mathscr{A}_0}\| \leqq 2^{\|T\|}$. $\|S_1(\mathscr{A}_0)\| \leqq 2^{\|T\|}$ and it is clear that $\|\{\mathscr{A}_0; \mathscr{A}_0 \subseteq \mathscr{A},\ A_0 \ \text{finite}\}\| = \|A\|$. Therefore, for any $\mathscr{A} \subseteq K(T)$, $\|S_1(\mathscr{A})\| \leqq 2^{\|T\|}\|A\|$, and T is superstable.

3) There exists an ordinal α such that, for any $p \in S^*(T)$, if $R(p) > \alpha$, then $R(p) = \infty$: let $\mathscr{M}$ be an $\aleph_0$-saturated model. If $p \in S^*(T)$, there is $\mathscr{A} \in K(T)$, A finite, such that $R(p) = R(p \restriction \mathscr{A})$, and $\mathscr{A}' \subseteq \mathscr{M}$ and $p' \in S_n(\mathscr{A}')$ such that $R(p') = R(p)$. There also exists $q \in S_n(\mathscr{M})$ such that $R(q) = R(p)$. Therefore it suffices to take $\alpha = \sup\{R(p); p \in \cup_n S_n(\mathscr{M})$ and $R(p) \in On\}$.

4) It should be noted that all rank-notions which have been introduced (R_0 ,

and the various notions introduced by Shelah) satisfy a stronger axiom than 5), namely;

5′) If $p \in S_n(\mathscr{A})$ and $R(p) \leqq \alpha$, there is $p_0 \subseteq p$, p_0 finite such that, for any $q \supseteq p_0$, $q \in S_n(\mathscr{A})$, $R(q) \leqq \alpha$.

This can be topologically expressed by a continuity property of R: For any $\mathscr{A}, n$, and $\alpha \in On$, the set $\{p; p \in S_n(\mathscr{A})$ and $R(p) > \alpha\}$ is closed in $S_n(\mathscr{A})$.

5. The rank U and Lachlan's theorem

In this section T will be assumed to be superstable, and L without constants or function symbols.

DEFINITION 1. Let R be a rank-notion; we shall say that R is *connected* if and only if:

1) For any $p \in S^*(T)$, $R(p) < \infty$.

2) For any $p \in S^*(T)$ and $\alpha \in On$, if $R(p) > \alpha$, then there is a complete extension p' of p such that $R(p') = \alpha$.

THEOREM 2. *There is one and only one connected rank-notion.*

PROOF. The proof will follow immediately from Lemma 3 and 4.

LEMMA 3. *Let V be a connected rank-notion and R be a rank-notion. Then for any $p \in S^*(T)$, $R(p) \geqq V(p)$.*

PROOF. By induction suppose that we know that for all $\beta < \alpha$ and $p \in S^*(T)$, $V(p) \geqq \beta$ implies $R(p) \geqq \beta$. Now let $q \in S_n(\mathscr{A})$ such that $V(p) \geqq \alpha$. Then for any $\beta < \alpha$, there exist $\mathscr{B} \supseteq \mathscr{A}$ and an extension q' of q in $S_n(\mathscr{B})$ such that $V(q') = \beta$. By the induction hypothesis we infer $R(q') \geqq \beta$, and by Theorem 4, 9, $R(q) > R(q')$ or $R(q) = \infty$. In either case $R(q) > \beta$, and since this is true for any $\beta < \alpha$, $R(q) \geqq \alpha$.

LEMMA 4. *There exists a connected rank-notion.*

PROOF. Let R be a rank-notion such that for any $p \in S^*(T)$, $R(p) < \infty$. Define a predicate on $On \times S^*(T)$, denoted by "$U(p) \geqq \alpha$", by induction on α:

"$U(p) \geqq 0$" is true for any $p \in S^*(T)$.

If α is a limit ordinal, then "$U(p) \geqq \alpha$", is true if and only if for all $\beta < \alpha$, "$U(p) \geqq \beta$" is true.

"$U(p) \geqq \alpha + 1$" if and only if there is a complete extension p' of p such that "$U(p') \geqq \alpha$" is true and $R(p) \neq R(p')$.

If there are ordinals α such that "$U(p) \geqq \alpha + 1$" is false, we shall denote $U(p)$ the least such ordinal; if not we set $U(p) = \infty$.

Claim 1. If $\alpha \geqq \beta$, and "$U(p) \geqq \alpha$" "is true, then "$U(p) \geqq \beta$" is true. The proof is by induction on β. For $\beta = 0$ or β a limit ordinal the result is clear. When α is a limit ordinal, it is again obvious, so we may assume that $\alpha = \alpha' + 1$ and $\beta = \beta' + 1$. Then there is p', a complete extension of p, such that $R(p') < R(p)$, and "$U(p') \geqq \alpha'$" is true. By the induction hypothesis we know that "$U(p') \geqq \beta'$" is true, and therefore "$U(p) \geqq \beta$" is true.

It easily follows that "$U(p) \geqq \alpha$" is true if and only if $U(p) \geqq \alpha$.

Claim 2. The map U verifies axioms 1/ and 2/ of Definition 4,1. This should be clear from the definition of U.

Claim 3. Suppose that $p \in S_n(\mathscr{B})$, $\mathscr{A} \subseteq \mathscr{B}$, and $R(p) = R(p \restriction \mathscr{A})$; then $U(p) = U(p \restriction \mathscr{A})$. We prove by induction on α that $U(p \restriction \mathscr{A}) \geqq \alpha$ implies $U(p) \geqq \alpha$. For $\alpha = 0$ and α limit there is nothing to say. Suppose $\alpha = \beta + 1$; there is $\mathscr{C} \supseteq \mathscr{A}$, $p' \in S_n(\mathscr{C})$ such that $U(p') \geqq \beta$ and $R(p') < R(p)$. We may suppose that $C - A$ is finite and $\mathscr{C}$ and $\mathscr{B}$ are included in a common $\| B \|^+$-saturated model $\mathscr{M}$. Let $\bar{c}'$ be a sequence which enumerates $C - A$, $\bar{d} \in M^n$ be such that $t(\bar{d}, \mathscr{B}) = p$, and $\bar{c} \in \bar{M}$, $\bar{d}' \in \bar{M}$ such that:

$$t(\bar{d}', \mathscr{C}) = p'$$

$$t(\bar{c} {}^\wedge \bar{d}, \mathscr{A}) = t(\bar{c}' {}^\wedge \bar{d}', \mathscr{A})$$

$$R(\bar{c}, \mathscr{A} \cup \bar{d}) = R(\bar{c}, \mathscr{B} \cup \bar{d}).$$

We then have:

$$U(\bar{d}, \mathscr{A} \cup \bar{c}) \geqq \beta \quad \text{and} \quad R(\bar{d}, \mathscr{A} \cup \bar{c}) < R(\bar{d}, \mathscr{A}) = R(\bar{d}, \mathscr{B}).$$

From the reciprocity principle, we have that for any $\bar{b} \in \bar{B}$,

$$R(\bar{b}, \mathscr{A}) = R(\bar{b}, \mathscr{A} \cup \bar{d}) \quad \text{and} \quad R(\bar{b}, \mathscr{A} \cup \bar{d}) = R(\bar{b}, \mathscr{A} \cup \bar{d} \cup \bar{c})$$

and therefore

$$R(\bar{b}, \mathscr{A} \cup \bar{c} \cup \bar{d}) = R(\bar{b}, \mathscr{A} \cup \bar{c}) \quad \text{and} \quad R(\bar{d}, \mathscr{B} \cup \bar{c}) = R(\bar{d}, \mathscr{A} \cup \bar{c}).$$

By the induction hypothesis, $U(\bar{d}, \mathscr{B} \cup \bar{c}) \geqq \beta$, and on the other hand $R(\bar{d}, \mathscr{B} \cup \bar{c}) < R(\bar{d}, \mathscr{B})$. Therefore, $U(\bar{d}, \mathscr{B}) = U(p) \geqq \beta + 1$.

It is also clear from the definition of U that if $U(p) = U(p \restriction \mathscr{A})$, then $R(p) = R(p \restriction \mathscr{A})$; hence U verifies axioms 3/, 4/ and 5/ of Definition 4,1: we have proved that U is a rank-notion.

Claim 4. For all $p \in S^*(T)$, $U(p)$ is an ordinal. Suppose not, and let $p \in S^*(T)$ be such that $U(p) = \infty$ and $R(p)$ is minimal. We know that there is an $\alpha \in On$ such that $U(q) \geqq \alpha$ implies $U(q) = \infty$. But $U(p) \geqq \alpha + 1$, and there is a complete extension p' of p such that $U(p') \geqq \alpha$ and $R(p) > R(p')$. But then $U(p') = \infty$, which contradicts the minimality of $R(p)$.

Claim 5. U is a connected rank-notion. The only thing which remains to be shown is that if $p \in S_n(\mathscr{A})$ and $U(p) > \beta$, then there is $\mathscr{B} \supseteq \mathscr{A}$ and $p' \in S_n(\mathscr{B})$ such that $U(p') = \beta$ and $p \subseteq p'$. Let

$$\beta_0 = \min\{U(q); q \quad \text{is a complete extension of } p \text{ and} \quad U(q) \geqq \beta\}.$$

Clearly $\beta_0 \geqq \beta$, and there is p_1, a complete extension of p, such that $U(p_1) = \beta_0$. If we suppose $\beta_0 > \beta$, then $U(p_1) \geqq \beta + 1$, and there is p_2 a complete extension of p_1 (and also of p) such that $U(p_2) \geqq \beta$ and $R(p_2) < R(p_1)$. But, by Theorem 4, 9, this implies $U(p_2) < U(p_1)$, and this contradicts the minimality of β_0.

This concludes the proof of Theorem 2. We see that the (unique) connected rank-notion U introduced in the above proof enjoys another universal property: it is the least rank-notion.[(†)]

We refer to [18, p. 367] or to [7, p. 80] for the definition of the *natural sum* of ordinals, which we shall denote by $\alpha(+)\alpha'$ (in [7], it is written $\sigma(\alpha, \alpha')$). A characteristic property of this natural sum is that, if

$$\alpha = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k$$

and

$$\alpha' = \omega^{\beta_1} n'_1 + \omega^{\beta_2} n'_2 + \cdots + \omega^{\beta_k} n'_k$$

where $n_1, n_2, \cdots, n_k, n'_1, \cdots, n'_k$ are non-negative integers and $(\beta_1, \beta_2, \cdots, \beta_k)$ is a strictly decreasing sequence of ordinals, then

$$\alpha(+)\alpha' = \omega^{\beta_1} \cdot (n_1 + n'_1) + \omega^{\beta_2} \cdot (n_2 + n'_2) + \cdots + \omega^{\beta_k} \cdot (n_k + n'_k).$$

We have

PROPOSITION 5. *For any ordinals* α, β, γ, α_1:

1) $\alpha(+)\beta = \beta(+)\alpha$ *and* $(\alpha(+)\beta)(+)\gamma = \alpha(+)(\beta(+)\gamma)$.

2) *If* $\alpha_1 < \alpha$, *then* $\alpha_1(+)\beta < \alpha(+)\beta$.

3) $\alpha(+)\beta \geqq \alpha + \beta$.

4) *If* $\alpha(+)\beta > \gamma$, *then there exist* α_2 *and* β_2 *such that* $\alpha_2 < \alpha$ *and* $\alpha_2(+)\beta \geqq \gamma$ *or* $\beta_2 < \beta$ *and* $\alpha(+)\beta_2 \geqq \gamma$.

(†) It was proved by Prof. Shelah that this rank does not in general satisfy axiom 5′ which was discussed at the end of the preceding section.

5) *If* $n \in \omega$, *then* $\alpha(+)n = \alpha + n$.

6) *If* $\beta \neq 0$ *and* $\gamma < \omega^{\beta}$, *then* $\alpha + \omega^{\beta} > \alpha(+)\gamma$.

PROOF. Properties 1) to 4) are in [7]. Properties 5) and 6) follow immediately from the characteristic property.

In Propositions 6 through 9, we shall assume $\mathcal{A} \subseteq \mathcal{M}$, $\bar{b} \in M^k$, $\bar{c} \in M^1$, and $\mathcal{M}$ $\| A \|^+$-saturated.

THEOREM 6. *Let* α *be an ordinal; then*:
$U(\bar{b}, \mathcal{A}) \geqq U(\bar{b}, \mathcal{A} \cup \bar{c})(+)\alpha$ *implies* $U(\bar{c}, \mathcal{A}) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b}) + \alpha$.

PROOF. The proof will proceed by induction on $U(\bar{b}, \mathcal{A})$. Let $\alpha' < \alpha$; then $U(\bar{b}, \mathcal{A}) > U(\bar{b}, \mathcal{A} \cup \bar{c})(+)\alpha'$, and since U is connected there is $\bar{d} \in \bar{M}$ such that

(1) $U(\bar{b}, \mathcal{A}) > U(\bar{b}, \mathcal{A} \cup \bar{d})$ and

(2) $U(\bar{b}, \mathcal{A} \cup \bar{d}) \geqq U(\bar{b}, \mathcal{A} \cup \bar{c})(+)\alpha'$.

On the other hand $t(\bar{b}^\wedge \bar{d}, \mathcal{A})$ is all that matters, so we may assume

(3) $U(\bar{d}, \mathcal{A} \cup \bar{b} \cup \bar{c}) = U(\bar{d}, \mathcal{A} \cup \bar{b})$,

and by reciprocity

(4) $U(\bar{c}, \mathcal{A} \cup \bar{b}) = U(\bar{c}, \mathcal{A} \cup \bar{b} \cup \bar{d})$.

Two cases arise:

Case 1. $U(\bar{b}, \mathcal{A} \cup \bar{c}) = U(\bar{b}, \mathcal{A} \cup \bar{c} \cup \bar{d})$.

Then by reciprocity, (3) and (1) we have

(5) $U(\bar{d}, \mathcal{A} \cup \bar{c}) = U(\bar{d}, \mathcal{A} \cup \bar{c} \cup \bar{b}) = U(\bar{d}, \mathcal{A} \cup \bar{b}) < U(\bar{d}, \mathcal{A})$.

Now since $U(\bar{b}, \mathcal{A} \cup \bar{d}) < U(\bar{b}, \mathcal{A})$, we can use (2) and utilize the induction hypothesis to get:

(6) $U(\bar{c}, \mathcal{A} \cup \bar{d}) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b} \cup \bar{d}) + \alpha'$.

By (4)

(7) $U(\bar{c}, \mathcal{A} \cup \bar{d}) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b}) + \alpha'$

and by (5) and reciprocity

(8) $U(\bar{c}, \mathcal{A}) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b}) + \alpha' + 1$.

Case 2. $U(\bar{b}, \mathcal{A} \cup \bar{c}) \geqq U(\bar{b}, \mathcal{A} \cup \bar{c} \cup \bar{d})$.

Then by (2)

(9) $U(\bar{b}, \mathcal{A} \cup \bar{d}) \geqq U(\bar{b}, \mathcal{A} \cup \bar{c} \cup \bar{d})(+)(\alpha' + 1)$

and, again by induction

(10) $U(\bar{c}, \mathcal{A} \cup d) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b} \cup \bar{d}) + \alpha' + 1$

and then, by (4), we get again

(8) $U(\bar{c}, \mathcal{A}) \geqq U(\bar{c}, \mathcal{A} \cup \bar{b}) + \alpha' + 1$.

So (8) is true in every case, for any $\alpha' < \alpha$, and

$$U(\bar{c}, \mathscr{A}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b}) + \alpha.$$

COROLLARY 7. *For any $\alpha \in On$ and $n \in \omega$, we have*

$$U(\bar{b}, \mathscr{A}) \geqq U(\bar{b}, \mathscr{A} \cup \bar{c}) + \omega^{\alpha} \cdot n \text{ iff } U(\bar{c}, \mathscr{A}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b}) + \omega^{\alpha} \cdot n.$$

PROOF. By Proposition 4, for any $\beta < \omega^{\alpha} \cdot n$, we have

$$U(\bar{b}, \mathscr{A}) \geqq U(\bar{b}, \mathscr{A} \cup \bar{c})(+)\beta$$

and by the last theorem

$$U({}^{-}c, \mathscr{A}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b}) + \beta.$$

Therefore, if $\alpha \neq 0$, $\omega^{\alpha} \cdot n$ is a limit ordinal and $U(\bar{c}, \mathscr{A}) \geqq \sup\{U(\bar{c}, \mathscr{A} \cup \bar{b}) + \beta ; \beta < \omega^{\alpha} \cdot n\} = U(\bar{c}, \mathscr{A} \cup \bar{b}) + \omega^{\alpha} \cdot n$.

If $\alpha = 0$, then $n = \omega^{\alpha} \cdot n$, and it is immediate.

THEOREM 8. *We have*

$$U(\bar{c}, \mathscr{A} \cup \bar{b}) + U(\bar{b}, \mathscr{A}) \leqq U(\bar{b}^{\wedge}\bar{c}, \mathscr{A}) \leqq U(\bar{c}, \mathscr{A} \cup \bar{b})(+)U(\bar{b}, \mathscr{A}).$$

PROOF.

1) We first prove that

$$U(\bar{c}, \mathscr{A} \cup \bar{b}) = U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}).$$

If we suppose $U(\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) + 1$, then we deduce

$$U(\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b} \cup \bar{b} \cup \bar{c})(+)(U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) + 1)$$

and by Theorem 6,

$$U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b} \cup \bar{c}) + U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) + 1,$$

which is impossible.

If we now assume $U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b}) + 1$, then

$$U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{b}^{\wedge}\bar{c}, \mathscr{A} \cup \bar{b} \cup \bar{c})(+)(U(\bar{c}, \mathscr{A} \cup \bar{b}) + 1)$$

and by Theorem 6,

$$U(\bar{c}, \mathscr{A} \cup \bar{b}) \geqq U(\bar{c}, \mathscr{A} \cup \bar{b} \cup \bar{b} \cup \bar{c}) + U(\bar{c}, \mathscr{A} \cup \bar{b}) + 1$$

which is again impossible.

2) $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \geqq U(\bar{c},\mathscr{A}\cup\bar{b})(+)(U(\bar{b},\mathscr{A})+1)$ is impossible because it implies $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \geqq U(\bar{b}{}^\wedge\bar{c},\mathscr{A}\cup\bar{b})(+)(U(\bar{b},\mathscr{A})+1)$ and

$$U(\bar{b},\mathscr{A}) \geqq U(\bar{b},\mathscr{A}\cup\bar{b}\cup\bar{c}) + U(\bar{b},\mathscr{A})+1.$$

3) On the other hand, we have $U(\bar{b},\mathscr{A}) \geqq U(\bar{b},\mathscr{A}\cup\bar{b}\cup\bar{c})(+)\,U(\bar{b},\mathscr{A})$ and therefore $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \geqq U(\bar{b}{}^\wedge\bar{c},\mathscr{A}\cup\bar{b})(+)\,U(\bar{b},\mathscr{A})$.

It is not in general the case that $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) = U(\bar{b},\mathscr{A}\cup\bar{c})(+)\,U(\bar{c},\mathscr{A})$. However, something positive can be said:

We shall say that $\bar{b}$ and $\bar{c}$ are *independent* over $\mathscr{A}$ if $|\bar{b}|$ and $|\bar{c}|$ are, and that $\bar{b}$ and $\bar{c}$ are independent if they are independent over the empty set.

PROPOSITION 9. *If $\bar{b}$ and $\bar{c}$ are independent over $\mathscr{A}$, then*

$$U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) = U(\bar{b},\mathscr{A})(+)\,U(\bar{c},\mathscr{A}).$$

PROOF. We only have to prove $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \geqq U(\bar{b},\mathscr{A})(+)\,U(\bar{c},\mathscr{A})$. We proceed by induction on $U(\bar{b},\mathscr{A})(+)\,U(\bar{c},\mathscr{A})$. Suppose by way of contradiction, that

$$U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) < U(\bar{b},\mathscr{A})(+)\,U(\bar{c},\mathscr{A});$$

then there exist α and β such that

(1) $\alpha < U(\bar{b},\mathscr{A})$ and $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \leqq \alpha\,(+)\,U(\bar{c},\mathscr{A})$ or

(2) $\beta < U(\bar{c},\mathscr{A})$ and $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}) \leqq \beta(+)\,U(\bar{b},\mathscr{A})$.

The proof is the same in either case, so we may assume (1).

There is $\bar{d} \in \bar{M}$ such that

(3) $U(\bar{b},\mathscr{A}\cup\bar{d}) \geqq \alpha$ and

(4) $U(\bar{b},\mathscr{A}) > U(\bar{b},\mathscr{A}\cup\bar{d})$.

Moreover, since nothing but $t(\bar{b}{}^\wedge\bar{d},\mathscr{A})$ matters, we may assume

(5) $U(\bar{d},\mathscr{A}\cup\bar{b}) = U(\bar{d},\mathscr{A}\cup\bar{b}\cup\bar{c})$.

Recall that by hypothesis

(6) $U(\bar{c},\mathscr{A}\cup\bar{b}) = U(\bar{c},\mathscr{A})$.

By (5) and (6) and reciprocity

(7) $U(\bar{c},\mathscr{A}\cup\bar{b}) = U(\bar{c},\mathscr{A}\cup\bar{b}\cup\bar{d}) = U(\bar{c},\mathscr{A}) = U(\bar{c},\mathscr{A}\cup\bar{b})$

and therefore $\bar{b}$ and $\bar{c}$ are independent over $\mathscr{A}\cup\bar{d}$. But since $U(\bar{b},\mathscr{A}\cup\bar{d})$ $(+)\,U(\bar{c},\mathscr{A}\cup\bar{d}) < U(\bar{b},\mathscr{A})(+)\,U(\bar{c},\mathscr{A})$ by (4) we may apply the induction hypothesis, and get

(8) $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}\cup\bar{d}) = U(\bar{b},\mathscr{A}\cup\bar{d})(+)\,U(\bar{c},\mathscr{A}\cup\bar{d})$

and from (3) and (7)

(9) $U(\bar{b}{}^\wedge\bar{c},\mathscr{A}\cup\bar{d}) \geqq \alpha(+)\,U(\bar{c},\mathscr{A})$.

From (5)

(10) $U(\bar{d}, \mathscr{A} \cup \bar{b} \,\hat{}\, \bar{c}) \leqq U(\bar{d}, \mathscr{A} \cup \bar{b}) < U(\bar{d}, \mathscr{A})$,

so by reciprocity

(11) $U(b \,\hat{}\, c, \mathscr{A} \cup \bar{d}) < U(b \,\hat{}\, c, \mathscr{A})$.

Now (9) and (11) contradict (1).

Definition 10. Let $\mathscr{E}$ be a set of substructures of M and $\mathscr{A} \subseteq \mathscr{M}$; we say that $\mathscr{E}$ is *independent* over $\mathscr{A}$ if for any $\mathscr{B} \in \mathscr{E}$, $\mathscr{B}$ and $\cup(\mathscr{E} - \{\mathscr{B}\})$ are independent over $\mathscr{A}$.

If S is a subset of $\bar{M}$, S is *independent* over $\mathscr{A}$ if $\{|\bar{s}|, -\bar{s} \in S\}$ is. If $\mathscr{A} = \varnothing$, we do not mention it.

We leave the proof of the two following propositions to the reader.

Proposition 11. *Let $\mathscr{A} \subseteq \mathscr{M}$, and for every $\alpha < \lambda$, $\mathscr{A}_\alpha \subseteq \mathscr{M}$ such that $\mathscr{A}_\alpha$ and $\bigcup_{\beta < \alpha} \mathscr{A}_\beta$ are independent over $\mathscr{A}$. Then $\{\mathscr{A}_\alpha; \alpha < \lambda\}$ is independent over $\mathscr{A}$.*

Proposition 12. *Let $\mathscr{A} \subseteq \mathscr{M}$ and $\mathscr{E}$ be a set of subsets of $\mathscr{M}$ which is independent over $\mathscr{A}$. Suppose that $\{\mathscr{E}_i; i \in I\}$ is a partition of $\mathscr{E}$. Then $\{\cup \mathscr{E}_i; i \in I\}$ is independent over $\mathscr{A}$.*

Theorem 13. *Let $\mathscr{A} \subseteq \mathscr{M}$, $\bar{b} \in \bar{M}$, $S \subseteq \bar{M}$, and suppose that S is independent over $\mathscr{A}$ and for any $\bar{s} \in S$, $U(\bar{b}, \mathscr{A} \cup \bar{s}) < U(\bar{b}, \mathscr{A})$. We may write*

(1) $U(\bar{b}, \mathscr{A}) = \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \cdots + \omega^{\beta_k} \cdot n_k$ where $k, n_1, \cdots, n_k$ are strictly positive integers, and $(\beta_1, \beta_2, \cdots, \beta_k)$ is a strictly decreasing sequence of ordinals. Then

$$\| S \| < (n_1 + 1) \quad (n_2 + 1) \cdots (n_k + 1).$$

Proof. We shall prove by induction on i, $0 \leqq i \leqq k$, that if the hypotheses are satisfied, and if

$$\| S \| = (n_{k-i} + 1)(n_{k-i+1} + 1) \cdots (n_k + 1), \quad \text{then for} \quad C = \cup \{|\bar{s}|;\ \bar{s} \in S\},$$

we have

$$U(b, \mathscr{A} \cup \mathscr{C}) \leqq \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \cdots + \omega^{\beta_{k-i-1}} \cdot n_{k-i-1}.$$

1) For $i = 0$, we have $\| S \| = n_k + 1$; let

$$S = \{s_j; 0 \leqq \bar{s}_j \leqq n_k\}.$$

Then, for any j, $0 \leqq j \leqq n_k$, we have

$$U(\bar{b}, \mathscr{A}) > U(\bar{b}, \mathscr{A} \cup \bar{s}_j).$$

From this and from (1) we infer:

$$U(\bar{b}, \mathscr{A}) \geqq U(\bar{b}, \mathscr{A} \cup \bar{s}_j) + \omega^{\beta_k}$$

and by Corollary 7, since s is independent over $\mathscr{A}$,

$$U(\bar{s}_j, \mathscr{A} \cup \bigcup_{i<j} |\bar{s}_i| \cup \bar{b}) + \omega^{\beta_k} \leqq U(\bar{s}_j, \mathscr{A} \cup \bar{b}) + \omega^{\beta_k}$$

$$\leqq U(\bar{s}_j, \mathscr{A}) = U(\bar{s}_j ; \mathscr{A} \cup \bigcup_{i<j} |\bar{s}_i|)$$

and again by Corollary 7,

$$U\Big(\bar{b}, \mathscr{A} \cup \bigcup_{i<j} |\bar{s}_i|\Big) \geqq U\Big(\bar{b}, \mathscr{A} \cup \bigcup_{i<j+1} |\bar{s}_i|\Big) + \omega^{\beta_k}.$$

This being true for j between 0 and n_k, we get

$$U(\bar{b}, \mathscr{A}) \geqq U(\bar{b}, \mathscr{A} \cup \mathscr{C}) + \omega^{\beta_{k+1}} \cdot (n_k + 1)$$

and by (1)

$$U(\bar{b}, \mathscr{A} \cup \mathscr{C}) < \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \cdots + \omega^{\beta_{k-1}} \cdot n_{k-1}.$$

2) Let us prove now the property for i $(0 < i \leqq k)$ assuming it for $i - 1$. Let $\{S_j ; 0 \leqq j \leqq n_{k-i}\}$ be a partition of S, such that for every j, $0 \leqq j \leqq n_{k-i}$,

$$\| S_j \| = (n_{k-i+1} + 1)(n_{k-i+2} + 1) \cdots (n_k + 1)$$

and set $\mathscr{C}_j = \cup \{|\bar{s}|, \bar{s} \in S_j\}$, and $\bar{s}_j$ a finite sequence which enumerates C_j. By Proposition 11, the set $\{\bar{s}_j ; 0 \leqq j \leqq n_{k-i}\}$ is independent over $\mathscr{A}$, and by the induction hypothesis, for any j, $0 \leqq j \leqq n_{k-i}$:

$$U(\bar{b}, \mathscr{A} \cup \bar{s}_j) < \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \cdots + \omega^{\beta_{k-i}} \cdot n_{k-i}.$$

As above

$$U(\bar{b}, \mathscr{A}) \geqq U(\bar{b}, \mathscr{A} \cup \bar{s}_j) + \omega^{\beta_{k-i}}$$

and

$$U\Big(\bar{s}_j, \mathscr{A} \cup \bigcup_{i<j} |\bar{s}_i| \cup \bar{b}\Big) + \omega^{\beta_{k-i}} \leqq U(\bar{s}_j, \mathscr{A} \cup \bar{b}) + \omega^{\beta_{k-i}}$$

$$\leqq U(\bar{s}_j, \mathscr{A}) = U\Big(\bar{s}_j, \mathscr{A} \cup \bigcup_{i<j} \bar{s}_i\Big)$$

and

$$U(b,\mathscr{A}) \geqq U(b,\mathscr{A}\cup\mathscr{C}) + \omega^{\beta_{k-i}}(n_{k-i}+1).$$

Hence from (1) we get:

$$U(b,\mathscr{A}\cup\mathscr{C}) < \omega^{\beta_1}\cdot n_1 + \omega^{\beta_2}\cdot n_2 + \cdots + \omega^{\beta_{k-i-1}}\cdot n_{k-i-1}$$

and the theorem is proved.

As a consequence we get the following theorem of Lachlan's ([8]):

THEOREM 14. *If T is a countable superstable theory which is not $\aleph_0$-categorical, then T has an infinite number of isomorphism types of countable models.*

PROOF. First of all, we may suppose that, for all $n \in \omega$, $S_n(T)$ is countable, and therefore for all $\mathscr{A}$ such that A is finite, $S_n(\mathscr{A})$ is countable, and there is a model prime over $\mathscr{A}$. On the other hand, since T is not $\aleph_0$-categorical, there is n and $p \in S_n(T)$ such that p is not isolated.

Consider the class:

$$K = \{\mathscr{M};\ \text{there exists } \mathscr{A} \subseteq \mathscr{M},\ A \text{ finite and } \mathscr{M} \text{ prime over } \mathscr{A}\}.$$

We shall prove that for any $\mathscr{M}_1, \mathscr{M}_2, \cdots, \mathscr{M}_k \in K$, there exists $\mathscr{M}' \in K$ which is not isomorphic to any $\mathscr{M}_i$, $1 \leqq i \leqq k$.

Let $\mathscr{M}''$ be an $\aleph_0$-saturated model of T; we may assume that each $\mathscr{M}_i$ is included in $\mathscr{M}''$; suppose $\bar{a}_i \in \bar{M}_i$, and $\mathscr{M}_i$ prime over $\bar{a}_i$; set $\bar{a} = \bar{a}_1{}^\wedge\bar{a}_2{}^\wedge\cdots{}^\wedge\bar{a}_k$, and

$$U(\bar{a}) = \omega^{\beta_1}\cdot n_1 + \omega^{\beta_2}\cdot n_2 + \cdots + \omega^{\beta_k}\cdot n_k$$

where $k, n_1, n_2, \cdots, n_k$ are strictly positive numbers and $(\beta_1, \beta_2, \cdots, \beta_k)$ a strictly decreasing sequence of ordinals. Set $m = (n_1+1)(n_2+1)\cdots(n_{k+1})$.

Now define by induction on $i \in \omega$, $\bar{b}_i \in M''^n$ such that $t(\bar{b}_i) = p$ and $U(\bar{b}_i, \bigcup_{j<i}\bar{b}_j) = U(\bar{b}_i)$. Then by Preposition 10, $\{\bar{b}_i, i \in \omega\}$ is independent. Let $\bar{b} = \bar{b}_0{}^\wedge\bar{b}_1{}^\wedge\cdots{}^\wedge\bar{b}_m$, and let $\mathscr{M}'$ be a model prime over $\bar{b}$. To prove that $\mathscr{M}'$ is not isomorphic to any $\mathscr{M}_i (0 \leqq i \leqq k)$ it suffices to prove that $t(\bar{b})$ is not realized in any $\mathscr{M}_i$, or more simply that $t(\bar{b})$ is not realized in $\mathscr{M}'_0$, a model prime over $\bar{a}$.

If we suppose the contrary, then there exist, for $0 \leqq i \leqq m$, $\bar{c}_i \in \bar{M}'_0$ such that $t(\bar{c}_i) = p$, and $\{\bar{c}_i; 0 \leqq i \leqq m\}$ is independent. But for all i, $0 \leqq i \leqq m$, $U(\bar{c}_i) > U(\bar{c}_i, \bar{a})$ by Corollary 4,16, as $t(\bar{c}_i)$ is not isolated while $t(\bar{c}_i, \bar{a})$ is; but this is impossible by the choice of m, the value of $U(\bar{a})$ and Theorem 13.

REFERENCES

1. J. T. Baldwin, *Countable theories categorical in some uncountable power*, Ph. D. Thesis, Simon Fraser University, 1971.

2. J. T. Baldwin and A. H. Lachlan, *On strongly minimal sets*, J. Symbolic Logic **36** (1971), 79–96.

3. D. Berthier, *Stability of non model complete theories, product, groups*, Centre Mathématique de l'École Polytechnique, Paris, 1973 (preprint).

4. D. Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970), 1–24.

5. C. C. Chang and H. J. Keisler, *Model Theory*, North Holland, Amsterdam, 1973.

6. M. Dickmann, *The problem of non finite axiomatic stability of $\aleph_1$-categorical theories*, The Proceedings of the Bertrand Rusell Memorial Logic Conference, 1971.

7. F. Hausdorff, *Set Theory*, Chelsea Publishing Co., 1962.

8. A. H. Lachlan, *On the number of countable models of a countable superstable theory*, Fund. Math. **81** (1974), 133–145.

9. D. Lascar, *Types definissables et produit de types*, C.R.A.S. Paris, **276** (1973), 1253–1256.

10. P. Lindström, *Remarks on some theorems of Keisler*, J. Symbolic Logic **33** (1968) 571–576.

11. J. Łos, *On the categoricity in power of elementary deductive systems*, Colloq. Math. **3** (1954), 58–62.

12. M. D. Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538.

13. M. D. Morley, *Countable models of $\aleph_1$-categorical theories*, Israel J. Math. **5** (1967), 65–72.

14. G. E. Sacks, *Saturated* Model Theory, Benjamin, Reading, Mass., 1972.

15. S. Shelah, *Stable theories*, Israel J. Math. **7** (1969), 187–202.

16. S. Shelah, *Stability and the f.c.p.; model-theoretic properties of formulas in first-order theories*, Ann. Math. Logic **3** (1971), 271–362.

17. S. Shelah, Manuscript of *Stable Theories*.

18. W. Sierpinski, *Cardinal and Ordinal Numbers*, Polish Scientific Publisher, Warsawa, 1965.

19. W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1955), 203–271.

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